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A variational approach of stationary Boltzmann equation under a condition of Poisson type

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Abstract

A functional is defined on a subset of $L^2(\Omega)$, with Ω bounded and so that the divergence theorem is valid. It shows that this functional is differentiable, coercive and weakly lower semi-continuous bound, and therefore has critical points which coincide with the solutions of the stationary Boltmann equation solutions under a condition of Poisson.

Key words: Boltzmann equitation, Kinetic Theory, existence critical points.

Un enfoque variacional de la ecuación de Boltzmann bajo una condición de Poisson

Resumen

Se define un funcional sobre un subconjunto de $L^2(\Omega)$, con Ω acotado y tal que el Teorema de la Divergencia sea válido. Se prueba que este funcional es diferenciable, coercivo y débilmente semicontinuo inferiormente y por tanto tiene puntos críticos que coinciden con las soluciones de las soluciones de la ecuación estacionaria de Boltmann bajo una condición de Poisson.

Palabras clave: ecuación de Boltzmann, Teoría Cinética, existencia de puntos críticos.

1. Introduction

Let us consider the following problem: Find $f(x,v) \geq 0$, $f \in L^2(\Omega)$ such that

$$\begin{cases} v \cdot \nabla_x f = Q(f,f), \; x \in \Omega, \; v \in \mathbb{R}^n \\ (vf \cdot x)\Delta_x(g) = Q(f,f)(g), \, x \in \Omega, v \in \mathbb{R}^n, \, g \in L^2(\Omega) \\ f(x,v) = e^{-|v|^2}, \, x \in \partial\Omega, v \in \mathbb{R}^n \end{cases} \quad (1)$$

$$Q(f,f) = \int_{\mathbb{R}^n} \int_{|w|=1} [w \cdot (v-u)] w [f(u')f(v') - f(u)f(v)] du dw$$

it is the collisions operator, being

$$\begin{aligned} v' &= v - [w \cdot (v-u)]w \\ u' &= u + [w \cdot (v-u)]w \end{aligned} \quad (2)$$

u, v, u' and v' are speeds precollision and postcollision respectively and we suppose that $u, v \in B_R(0)$ in $\mathbb{R}^n$, therefore $u, v' \in B_{3R}(0) \subseteq \mathbb{R}^n$ and every velocity is contained in $B_1(0)$, w is a unitary vector, here we use the notation $f(u)=f(x,u)$, $f(v)=f(x,v)$ etc., and $(g \cdot (v-u))$ it is the kernel of Collision operator. We consider the solution in the space $H = \{f \in L^2(\Omega): \nabla_x f \in L^2(\Omega)\}$ with norm $\|f\|_H^2 = \|f\|_{L^2(\Omega)}^2 + \|\nabla_x f\|_{L^2(\Omega)}^2$, and $\|f\|_{L^2(\Omega)}^2 = \int_\Omega |f|^2 dx$.

In physics and applications, it is important to consider the flows of fluid (or gas) and clouds of particles outside and particles inside obstacles. Typical examples are the airfoil problem and the re-entry problem for spaces shuttles [1].

The classical Boltzmann equation is a mathematical model whose describes a gas as a collection of particles moving in a three-dimensional space. The solution is a function $f(t, x, v) \geq 0$, $t \in \mathbb{R}$, $x \in \mathbb{R}^3$, $v \in \mathbb{R}^3$. Here $f \to 0, |v| \to 0$. f is called velocity distribution function. In this paper we study the stationary Boltzmann equation (i.e., f does not depend of t) in bounded domain.

L^1 Solutions for the Non-linear stationary Boltzmann equation have been obtained by techniques of weak Compactness. Examples of results of existence close to equilibrium for stationary Povzner equation in bounded domains of $\mathbb{R}^n$ are obtained in [2, 3] and L^1 solutions for the Non-linear stationary Boltzmann equation in a slab is studied in [4, 5]. Also problems of semispace for Non-linear stationary Boltzmann equation in the slab with data can be resolved sometimes by these techniques, for a truncated collision operator for high speeds and small values of the component of velocity in the direction of the slab [6]. For bounded domains in $\mathbb{R}^n$ a result of existence was obtained in general for stationary Boltzmann equation under an additional truncation for small speeds to remove the cut-off of small velocities [7]. A Non-linear stationary Boltzmann equation with large boundary data is an open problem in spaces with dimensions greater than one. In this paper we present a variational approach of stationary Boltzmann equation; close to equilibrium in $\mathbb{R}^n$ the situation is clearer since the technique of contraction mappings exists and here there are many results of existence [8-15]. We develop four lemmas for the result of existence.

2. Preliminary

We give some basic definitions about variational methods (see [17-21]).

Definition 1. Let H be a Banach space. A functional J is a function defined on H, or on some subspace of H, with values in$\mathbb{R}$.

Definition 2. A functional $J \in H \to \mathbb{R}$ on Banach space H is called coercive if and only if $\lim_{\|u\| \to +\infty} J(u) = +\infty$.

Definition 3. A functional $J: H \to \mathbb{R}$ on Banach space H is called weakly lower semi-continuous (w.l.s.c.) if and only if for every sequence $u_n \in H$ such that $u_n \rightharpoonup u$ one has that $J(u) \leq \liminf J(u_n)$.

Remark. Let H and F Banach spaces. We denote by $L(H,K)$ the space of linear continuous application from H into F, equipped with the norm $\|T\|_{L(H,K)} = \sup\{\|Tx\|_H : \|x\|_k \leq 1\}, T \in L(H, K)$.

Definition 4. A functional $J \in H \to \mathbb{R}$ on Banach space H is Fréchet differentiable at $u \in U \subseteq H$ (U is an open set), whit derivative $dJ(u) \in L(H, \mathbb{R})$ if $J(u + h) = J(u) + dJ(u)[h] + r(h)$, where $\lim_{h \to 0} \frac{r(h)}{\|h\|} = 0$. J. is said Fréchet differentiable on U if it is Fréchet differentiable at every point $u \in U$. We will often use $J'(u)$ instead of $dJ(u)$.

Definition 5. A critical point of a functional $J: H \to \mathbb{R}$ is a $z \in H$ such that J is Fréchet differentiable at z and $J'(z)=0$.

Definition 6. A functional $J \in H \to \mathbb{R}$ on Banach space H is Gâteaux differentiable at $u \in U \subseteq H$ (U is an open set) in the direction $v \in$ H if there exists $\lim_{t \to 0} \frac{J(u + tv) - J(u)}{t}$ for all $v \in H$. We use the notation $J'(u)v$ for this limit. $J'(u)v$ is called Gâteaux derivative.

Now we present the previous results to prove the existence of solutions of the stationary Boltzmann equation.

Lemma 1. Let $\Omega \subseteq \mathbb{R}^n$, a bounded domain such that given $x_0 \in \partial\Omega$ there a neighborhood U de x_0 on $\mathbb{R}^n$ and a function $\varphi: U \to \mathbb{R}$ continuously differentiable so that

i) $\nabla_x \varphi(x) \neq 0$ si $x \in U$

ii) $\partial\Omega \cap U = \{x \in U : \varphi(x) = 0\}$

iii) $\Omega \cap U = \{x \in U : \varphi(x) < 0\}$ and

(For this class of sets, the divergence theorem is true applied to continuously differentiable fields [16]), $div_x(\cdot) \in L^2(\cdot) \in L^2(\Omega)$.

If

$$J(f) := \int_\Omega \int_{B_{3R}(0)} vf \cdot \nabla_x f dv dx \qquad (3)$$

then

$$J'(f)g = \int_\Omega \int_{B_{3R}(0)} (v \cdot \nabla_x f - Q(f, f)) g dv dx \ c.t.p. en \, \Omega$$

for all $g \in H$.

Proof. $J'(f)g = \lim_{h\to 0} \frac{1}{h}[J(f+h) - J(f)]$, and applying (3)

$$J'(f)g = \int_\Omega \int_{B_{3R}(0)} v \cdot \nabla_x f g dv dx + \int_\Omega \int_{B_{3R}(0)} vf \cdot \nabla_x g dv dx.$$

Since if $g \in C^1(\Omega) \cap C^2(\Omega)$, we get

$$div_x[v \cdot x)\nabla_x g] = v \cdot \nabla_x g + (v \cdot x)\Delta_x g,$$

then

$$f div_x[(v \cdot x)\nabla_x g] = f(v \cdot \nabla_x g) + f(v \cdot x)\Delta_x g$$

Integrating, we have with $div_x(\cdot) \in L^2(\Omega)$, $\nabla_x g \in L^2(\Omega)$

$$\int_\Omega \int_{B_{3R}(0)} f div_x[(v \cdot x)\nabla_x g] dv dx = \int_\Omega \int_{B_{3R}(0)} f(v \cdot \nabla_x g) dv dx + \int_\Omega \int_{B_{3R}(0)} f(v \cdot x)\Delta_x g \, dv dx.$$

Applying the divergence theorem and Fubini theorem, we get

$$\int_{B_{3R}(0)} \left[\int_\Omega f div_x[(v \cdot x)\nabla_x g] dx\right] dv = 0$$

$$= \int_\Omega \int_{B_{3R}(0)} f(v \cdot \nabla_x g) dv dx + \int_\Omega \int_{B_{3R}(0)} f(v \cdot x)\Delta_x g \, dv dx$$

$$= \int_\Omega \int_{B_{3R}(0)} fv \cdot \nabla_x g dv dx + \int_\Omega \int_{B_{3R}(0)} (fv \cdot x)\Delta_x g \, dv dx$$

$$= \int_\Omega \int_{B_{3R}(0)} fv \cdot \nabla_x g dv dx + \int_\Omega \int_{B_{3R}(0)} Q(f, f) g \, dv dx$$

for all $g \in H$. Therefore

$$J'(f)g = \int_\Omega \int_{B_{3R}(0)} v \cdot \nabla_x f g dv dx - \int_\Omega \int_{B_{3R}(0)} Q(f, f) g \, dv dx$$

for all $g \in H$.

Lemma 2. If $f \in H, \Omega \subseteq \mathbb{R}^n$ with the same hypothesis of Lemma 1, then J is coercive.

Proof. Suppose that for all $M \geq 0$ we have, $\left|\int_\Omega \int_{B_{3R}(0)} vf \cdot \nabla_x f dv dx\right| \leq M$, then $0 \leq (vf + \nabla_x f)^2 = \|vf\|^2 + 2vf \cdot \nabla_x f + \|\nabla_x f\|^2$. This implies that

$$-\|vf\|^2 - \|\nabla_x f\|^2 \leq 2vf \cdot \nabla_x f \Leftrightarrow$$

$$-\|v\|^2 |f|^2 - \|\nabla_x f\|^2 \leq 2vf \cdot \nabla_x f \Leftrightarrow$$

$$-\frac{1}{2}\|v\|^2 |f|^2 - \frac{1}{2}\|\nabla_x f\|^2 \leq vf \cdot \nabla_x f.$$

Hence

$$\int_\Omega \int_{B_{3R}(0)} -\frac{1}{2}\|v\|^2 |f|^2 dv dx - \int_\Omega \int_{B_{3R}(0)} \frac{1}{2}\|\nabla_x f\|^2 dv dx \leq \int_\Omega \int_{B_{3R}(0)} vf \cdot \nabla_x f \, dv dx,$$

then

$$\left|-\int_\Omega \int_{B_{3R}(0)} -\frac{1}{2}\|v\|^2 |f|^2 dv dx - \frac{1}{2}\int_\Omega \int_{B_{3R}(0)} \|\nabla_x f\|^2 dv dx\right| \leq M;$$

$$\left|\int_\Omega \int_{B_{3R}(0)} \frac{1}{2}\|v\|^2 |f|^2 dv dx + \frac{1}{2}\int_\Omega \int_{B_{3R}(0)} \|\nabla_x f\|^2 dv dx\right| \leq M;$$

$$\frac{1}{2}\int_\Omega \int_{B_{3R}(0)} \|\nabla_x f\|^2 dv dx \leq M - \frac{1}{2}\int_\Omega \int_{B_{3R}(0)} \|v\|^2 |f|^2 dv dx;$$

$$\frac{1}{2}\int_\Omega \int_{B_{3R}(0)} \|\nabla_x f\|^2 dv dx + \frac{1}{2}\int_\Omega \int_{B_{3R}(0)} |f|^2 dv dx$$

$$\leq M + \frac{1}{2}\int_\Omega \int_{B_{3R}(0)} |f|^2 dv dx - \frac{1}{2}\int_\Omega \int_{B_{3R}(0)} \|v\|^2 |f|^2 dv dx;$$

$$\frac{1}{2}\int_{B_{3R}(0)} dv \int_\Omega \|\nabla_x f\|^2 dx + \frac{1}{2}\int_{B_{3R}(0)} dv \int_\Omega |f|^2 dx$$

$$\leq M + \frac{1}{2}\int_{B_{3R}(0)} dv \int_\Omega |f|^2 dx - \frac{1}{2}\int_{B_{3R}(0)} \|v\|^2 dv \int_\Omega |f|^2 dx;$$

$$\frac{1}{2}\int_{B_{3R}(0)} dv (\|f\|_H^2) \leq M + \frac{1}{2}\|f\|^2 \left(\int_{B_{3R}(0)} dv - \int_{B_{3R}(0)} \|v\|^2 dv\right)$$

$$\leq M + \frac{1}{2}\|f\|_H^2 \left(\int_{B_{3R}(0)} dv - \int_{B_{3R}(0)} \|v\|^2 dv\right);$$

$$\frac{1}{2}\|f\|_H^2 \int_{B_{3R}(0)} dv - \frac{1}{2}\|f\|_H^2 \left(\int_{B_{3R}(0)} dv - \int_{B_{3R}(0)} \|v\|^2 dv\right) \leq M.$$

Then

$$\|f\|_H^2 \leq \frac{M}{\frac{1}{2}\int_{B_{3R}(0)} \|v\|^2 dv}.$$

Therefore J coercive.

Lemma 3. J is Fréchet differentiable in $f \in H$.

Proof. We want to prove that there is a $J'(f) \in L(H,\mathbb{R})$ such that $J(f+g) = J(f) + J'(f)[g] + r(g)$, where $\lim_{h\to 0} \frac{r(g)}{\|g\|} = 0$. Indeed, consider $g \in H$, then:

$$J(f+g) - J(f) =$$

$$\int_\Omega \int_{B_{3R}(0)} v(f+g)\cdot\nabla_x(f+g)dvdx - \int_\Omega \int_{B_{3R}(0)} vf\cdot\nabla_x f dvdx$$

$$= \int_\Omega \int_{B_{3R}(0)} vf\cdot\nabla_x f dvdx + \int_\Omega \int_{B_{3R}(0)} vg\cdot\nabla_x f dvdx +$$

$$\int_\Omega \int_{B_{3R}(0)} vf\cdot\nabla_x g dvdx + \int_\Omega \int_{B_{3R}(0)} vg\cdot\nabla_x g dvdx -$$

$$\int_\Omega \int_{B_{3R}(0)} vf\cdot\nabla_x f dvdx$$

$$= \int_\Omega \int_{B_{3R}(0)} vg\cdot\nabla_x f dvdx + \int_\Omega \int_{B_{3R}(0)} vf\cdot\nabla_x g dvdx +$$

$$\int_\Omega \int_{B_{3R}(0)} vg\cdot\nabla_x g\, dvdx$$

$$= \int_\Omega \int_{B_{3R}(0)} (v\cdot\nabla_x f)gdvdx + \int_\Omega \int_{B_{3R}(0)} (vf\cdot\nabla_x g)dvdx +$$

$$\int_\Omega \int_{B_{3R}(0)} vg\cdot\nabla_x g\, dvdx.$$

We see that the term

$$\int_\Omega \int_{B_{3R}(0)} (v\cdot\nabla_x f)gdvdx + \int_\Omega \int_{B_{3R}(0)} (vf\cdot\nabla_x g)dvdx,$$

is a linear continuous application.

Now,

$$0 \le (\|\nabla_x g\| - \|vg\|)^2 = \|\nabla_x g\|^2 - 2\nabla_x g\cdot vg + \|vg\|^2;$$

$$2\nabla_x g\cdot vg \le \|\nabla_x g\|^2 + \|vg\|^2;$$

$$2\int_\Omega \int_{B_{3R}(0)} \nabla_x g\cdot vgdvdx \le \int_\Omega \int_{B_{3R}(0)} \|\nabla_x g\|^2 dvdx +$$

$$\int_\Omega \int_{B_{3R}(0)} \|vg\|^2 dvdx;$$

then

$$\frac{\int_\Omega \int_{B_{3R}(0)} \nabla_x g\cdot vgdvdx}{\|g\|} \le$$

$$\frac{\frac{1}{2}\int_\Omega \int_{B_{3R}(0)} \|\nabla_x g\|^2 dvdx + \frac{1}{2}\int_\Omega \int_{B_{3R}(0)} \|vg\|^2 dvdx}{\|g\|};$$

$$\frac{\int_\Omega \int_{B_{3R}(0)} \nabla_x g\cdot vgdvdx}{\sqrt{\int_\Omega \int_{B_{3R}(0)} \|g\|^2 dvdx + \int_\Omega \int_{B_{3R}(0)} \|\nabla_x g\|^2 dvdx}} \le$$

$$\frac{\frac{1}{2}\int_\Omega \int_{B_{3R}(0)} \|\nabla_x g\|^2 dvdx + \frac{1}{2}\int_\Omega \int_{B_{3R}(0)} \|vg\|^2 dvdx}{\sqrt{\int_\Omega \int_{B_{3R}(0)} \|g\|^2 dvdx + \int_\Omega \int_{B_{3R}(0)} \|\nabla_x g\|^2 dvdx}}.$$

The term of the right in the above inequality tends to zero if $\|g\|$ tends to zero, so

$$\frac{\int_\Omega \int_{B_{3R}(0)} vg\nabla_x g\, dvdx}{\|g\|} \to 0, \text{ if } \|g\| \to 0;$$

i.e., $\lim_{h\to 0} \frac{r(g)}{\|g\|} = 0$, where

$$r(g) = \int_\Omega \int_{B_{3R}(0)} vg\nabla_x g\, dvdx.$$

Therefore J is Fréchet differentiable.

Lemma 4. J is weakly lower semi-continuous.

Proof. Since J is coercive in H, for any M constant there exist $C(M)$ such that for all $f \in H$ with $|J(f)| \le M$ then $\|f_n\|_H \le C(M)$, let $f_n \rightharpoonup f$ with $\|f_n\|_H \le C(M)$, as J is continuous we have that $J(f_n) \le \liminf f_n$ and J is weakly lower semi-continuous.

3. Result

Theorem. Let $\Omega \subseteq \mathbb{R}^n$ with the same hypothesis of Lemma 1 and $J(u)$ defined in (3), u, v, u', v' belonging to $B_1(0)$ (unit ball), then J has a critical point which is solution of (1).

Proof. The previous lemmas and the Theorem 5.5 of [21] guarantee that there exist $z \in H$, such that z is a global minimum and $J'(z) \equiv 0$, i.e., is solution of (1).

4. Conclusion

In this article has been shown that a solution stationary Boltzmann equation in a bounded domain in $L^2(\Omega)$ exists. The existence has been proved through variational methods, defining a functional and showing that it is differentiable, coercive and weakly lower semi-continuous bound and that therefore has critical points which are solutions of the stationary Boltzmann equation.

Although, we've been achieved satisfactory results, we could try to seek solutions to the Boltzmann equation in general domains in $L^1(\Omega)$, as well as to reflect on whether the variational methods throw some result in this way and at the same time find methods that prove uniqueness of the solution.

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