On totally contact umbilical submanifolds of a manifold with a sasakian 3-structure

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Abstract

In our paper [5] we proved that any totally contact umbilical submanifold M of a manifold with a Sasakian 3-structure with dim µk ≥ 1, ∀x ∈ M, is totally contact geodesic. In the present paper we solve the remaining cases. Namely, when dim µk = 0, or dim µk = 1, M is totally contact geodesic or an intrinsic sphere respectively.

Key words: Sasakian-3 structure; totally contact umbilical; totally contact geodesic; extrinsic sphere.

Sobre subvariedades con contacto umbilical completo de una variedad con una estructura-3 sasakian

Resumen

En nuestro trabajo [5] probamos que cualquier subvariedad con contacto umbilical completo de una variedad con una estructura-3 Sasakian con dim µk ≥ 1, para todo x que pertenece a M, es de contacto geodésico total. En el presente trabajo resolvemos los casos restantes. A saber, cuando dim µk = 0 ó dim µk = 1, M es de contacto geodésico total o una esfera intrínseca, respectivamente.

Palabras clave: Estructura 3-Sasakian, contacto umbilical completo, contacto geodésico total, esfera extrínseca.

Introduction

It is well known [see [5]] that the tangent bundle \( TM \) of a semi-invariant submanifold \( M \) (called also contact CR-submanifold), tangent to the structure vector field \( \xi \), has the decomposition \( TM = D \oplus D^\perp \oplus \{ \xi \} \), where \( D \) and \( D^\perp \) are the invariant and anti-invariant distributions on \( M \), with respect to the structure tensor field \( f \) on manifold \( M \). Equivalently, \( M \) is a semi-invariant submanifold of a manifold \( M \) if its normal bundle \( TM^\perp \) has the decomposition \( TM^\perp = \mu \oplus \mu^\perp \), where \( \mu \) and \( \mu^\perp \) are invariant and anti-invariant sub-bundles of \( TM^\perp \) with respect to \( f \). The equivalence fails in the case of manifold with a Sasakian 3-structure. In this case the distribution \( D^\perp \) is not anti-invariant to the structure tensor field.
According to a known result (see [2]) a totally contact umbilical semi-invariant submanifold of a manifold with a Sasakian 3-structure with $\dim M = 3 > 1$, for any $x \in M$, is totally contact geodesic. The main purpose of the present paper is to study the remaining cases. More precisely we prove that $M$ is totally contact geodesic submanifold of $\tilde{M}$ if $\dim M = 1$. If $\dim M = 1$, $x \in M$, but $M$ is not totally contact geodesic, then $M$ is extrinsic sphere.

### Preliminaries

Let $\tilde{M}$ be a $(4n+3)$-dimensional differentiable manifold with an almost contact metric 3-structure $(\phi, \xi, \eta, g)$, $a \in \{1, 2, 3\}$. Then we have

- $f_a = -I + \eta_a \otimes \xi_a$,

- $f_a(\xi_a) = \delta_{ab}$,

- $f_a \circ f_a = -\eta_a \otimes \xi_a = -f_a \circ f_a + \eta_a \otimes \xi_a = f_a$,

- $f_a \circ f_b - \eta_a \otimes \xi_a = -f_a \circ f_a + \eta_a \otimes \xi_a = f_a$,

- $\eta_a(X) = g(X, \xi_a)$

for any cyclic permutation (a, b, c) of $\{1, 2, 3\}$, where $X$ and $Y$ are the vector fields tangent to $\tilde{M}$, $\delta$ is the Kronecker's delta. Then $\tilde{M}$ is called a manifold with a Sasakian 3-structure, if each $(f_a, \xi_a, \eta_a, g)$ is a Sasakian 3-structure, i.e. (see [6]):

- $\tilde{\nabla}_X f_a = g(X, Y)\xi_a - \eta_a(Y)X$, $a \in \{1, 2, 3\}$

for any vector fields $X$, $Y$ tangent to $\tilde{M}$ where $\tilde{\nabla}$ is the Levi-Civita connection on $\tilde{M}$. It is easy to see that $[\xi_a, \zeta_a] = 2\xi_a$ for any cyclic permutation (a, b, c) of $\{1, 2, 3\}$. Throughout the paper, all manifolds and maps are supposed differentiable of class $C^\infty$. We denote by $\pi^{\tilde{M}}$ the module of the differentiable functions on $M$ and by $\pi^E$ the module of smooth sections of a vector bundle $E$ over $M$. We use the same notations for any manifolds involved in the study.

The curvature tensor $K$ of $\tilde{M}$ is defined by

$$K(X, Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X, Y]}Z,$$

for all $X, Y, Z \in \Gamma(\tilde{TM})$.

Because the structure tensor field $f_a$ verifies (1.2a) then the curvature tensor field $K$ verify

- $a = \phi(K(X, Y)Z = f_a(K(X, Y)Z + g(f_aX, Z)Y - g(Y, Z))f_aX + g(X, Z))f_aY - g(f_aX, Z)X$,

- $b = g(K(f_aX, f_aY)Z)Z = g(K(X, Y)Z, W) - \eta_a(Y)\eta_a(Z)g(X, W) - \eta_a(X)\eta_a(W)g(Y, Z) + \eta_a(X)\eta_a(Z)g(Y, W) + \eta_a(Y)\eta_a(W)g(X, Z)$,

- $c = K(X, \xi_a)Y = \eta_a(Y)X - g(Y, Z)\xi_a$,

for any vector fields $X, Y, Z, W \in \Gamma(\tilde{TM})$ (1.3)

Now, let $M$ be a $m$-dimensional Riemannian manifold isometrically immersed in $\tilde{M}$, and suppose that the structure vector fields $\xi_1, \xi_2, \xi_3$ of $\tilde{M}$ are tangent to $M$. We denote by $TM$ and $TM^\perp$ the tangent bundle and the normal bundle to $M$, respectively. We also denote by $\{\xi\}$ the distribution spanned by $\xi_1, \xi_2, \xi_3$ on $M$. The induced metric tensor on $M$ will be denoted by the same symbol $g$.

The submanifold $M$ of a manifold with a Sasakian 3-structure is called semi-invariant submanifold (see [2]) if there exists a vector subbundle $\mu \subset TM^\perp$ such that

$$\phi|_M = \mu; f_a(\mu^\perp) \subset TM, \quad a \in \{1, 2, 3\},$$

where $\mu^\perp$ is the complementary orthogonal bundle to $\mu$ in $TM^\perp$. It is easy to see that any real hypersurface of $\tilde{M}$ is a semi-invariant submanifold. Next, denote $f_a|_M = D_a$, $a \in \{1, 2, 3\}$, $x \in M$. By using (1.1e) and (1.1f) it is obtained that

$$D_a = D_{a1} \oplus D_{a2} \oplus D_{a3},$$

where $D_{a1}, D_{a2}, D_{a3}$ are mutually orthogonal subspaces of $\pi_x$ and have the same dimension $\delta$. The dimension of $\tilde{T}_M$ and $M$. We note that the subspaces $D_{a1}, a \in \{1, 2, 3\}$ do not define in general a distribution on $M$, but the mapping.

$D^\perp: x \rightarrow D^\perp_x = D_{a1} \oplus D_{a2} \oplus D_{a3}$.
is a 3a-dimensional distribution on M (s = dim $\mu^*_a$). By straightforward calculation we deduce

$$f_a(D_{m}) = \mu^*_a; \quad f_\alpha(D_{m}) = D_{ce}$$  \quad (1.4)$$

for each $x \in M$, where $\{a, b, c\}$ is a cyclic permutation of $\{1, 2, 3\}$. We denote by D the complementary orthogonal distribution to $D^* \otimes \{\xi\}$ in $TM$. It follows that the distribution D is invariant with respect to the action of $f_1, f_2, f_3$, that is $f_a(D) = D$, $a \in \{1, 2, 3\}$. Thus M is semi-invariant submanifold of a manifold $\tilde{M}$ with a Sasakian 3-structure if

$$TM = D \otimes D^* \otimes \{\xi\}$$

where $D$, $\{\xi\}$ and $D^*$ are the above distributions. We note that $D^*$ is not anti-invariant distribution (see (1.4b)).

From the general theory of Riemannian submanifolds, recall the Gauss and Weingarten formulae

a) $\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y)$,

b) $\tilde{\nabla}_X N = -A_N X + \tilde{\nabla}_N X$,

$$\forall X, Y \in \Gamma(TM), \ N \in \Gamma(TM^*).$$  \quad (1.5)$$

where h is the second fundamental form of M, $A_N$ is the shape operator with respect to the normal section N, $\nabla$ and $\tilde{\nabla}$ are the induced connections by $\nabla$ on TM and $TM^*$ and $\tilde{\nabla}$, respectively. The Codazzi equation is given by

$$g((\nabla X Y) Z, N) - g((\tilde{\nabla} X h) Y, Z)g(\nabla X, Y) = -g(\nabla Y h) X, Z)g(\nabla X, Y),$$

$$\forall X, Y, Z \in \Gamma(TM), \ N \in \Gamma(TM^*).$$  \quad (1.6)$$

It is known that if $\{e_i\}$, $i = 1, ..., m$ is an orthonormal basis of $\Gamma(TM)$, then the mean curvature vector field of M, denoted by H, is given by

$$H = \frac{1}{m} \sum_{i=1}^{m} h(e_i, e_i).$$

The submanifold M is called totally contact umbilical if the second fundamental form h of M is expressed as follows

$$h(X, Y) = \sum_{a=1}^{3} \left(\eta_a X h(Y, \xi_a) + \eta_a Y h(X, \xi_a)\right), \ \forall X, Y \in \Gamma(TM)$$  \quad (1.7)$$

If $H = 0$ and (1.7) holds, then M is called totally contact geodesic submanifold of $\tilde{M}$.

It is known that any sphere of a Euclidean space is totally umbilical and has positive constant curvature. Also we recall that M is an extrinsic sphere of $\tilde{M}$ if it is totally contact umbilical and has parallel the mean curvature vector field $H \neq 0$, that is,

$$\nabla_X H = 0, \ \forall X \in \Gamma(TM).$$

Finally we recall some properties of semi-invariant submanifolds of a manifold $\tilde{M}$ with a Sasakian 3-structure, for later use (see [2]).

**Proposition. 1.1.** Let M be a semi-invariant submanifold of a manifold with a Sasakian 3-structure. Then

a) $h(X, \xi_\alpha) = 0$;

b) $h(Z, \xi_\alpha) = -f_a Z, \ \forall X \in \Gamma(D), \ Z \in \Gamma(f_a(D^*))$  \quad (1.8)$$

Also we see that if M is totally contact umbilical then

$$\nabla_X h(Y, Z) = 3g(Y, Z) \nabla_X H,$$  \quad (1.9)$$

if Y and Z belong to $\Gamma(D)$ and $X \in \Gamma(TM)$.

**Main Results**

Let M be a real m-dimensional submanifold of a 2n+1-dimensional manifold $\tilde{M}$ with a Sasakian 3-structure. It was proved [see [2]] that if M is totally contact umbilical semi-invariant proper submanifold (dim D > 0; dim $D^* > 0$), with $s = \dim \mu^*_a > 1$, $x \in M$ then M must be totally contact geodesic. Then it remains to study the cases $s = 0$ and $s = 1$. To this end we first prove the following general lemma.

**Lemma. 2.1.** Let M be a totally contact umbilical semi-invariant submanifold of a manifold $\tilde{M}$ with a Sasakian 3-structure and $D \neq \{0\}$. Then
the mean curvature vector field \( H \) of \( M \) is a global section of \( \Gamma(\mu^j) \).

**Proof.** Let \( X \in \Gamma(D) \) a unit vector field and \( N \in \Gamma(\mu) \). By using (1.1g), (1.2a), (1.5a) and (1.7) we deduce that

\[
g(H, N) = g(g(X, X)H, N) = g(\nabla_X X, N) = g(H, J_aX) = 0
\]

which proves our assertion.

Now we see that if \( s = 0 \), then \( H = 0 \) and \( M \) is totally contact geodesic. Next, because \( M \) is not totally contact geodesic and it is supposed to be connected, then let \( \alpha = [H] \neq 0 \). Denote

\[
a) \ U = \frac{1}{\alpha} H \quad \text{b) } W_a = f_aU, \ a \in \{1,2,3\}.
\]

**Lemma 2.2.** Let \( M \) be a totally contact umbilical semi-invariant submanifold of a manifold \( \tilde{M} \) with a Sasakian 3-structure. Then we have

\[
\nabla_{\dot{X}} H \in \Gamma(\mu^j), \ \forall \ X \in \Gamma(TM).
\]

**Proof.** Let \( X \in \Gamma(TM) \) and \( N \in \Gamma(\mu) \). Now by using Lemma 2.1 we have \( H \in \Gamma(\mu^j) \). By using (1.1g), (1.2a), (1.6b) and (1.7) we infer that

\[
g(\nabla_{\dot{X}} H, N) = g(\nabla_X f_a X - (\nabla_X J_a X) f_a N) = g(H, J_a X) = 0.
\]

Therefore our assertion is proved.

Now we prove the main result of the paper

**Theorem 2.1.** Let \( M \) be a proper totally contact umbilical semi-invariant submanifold of a manifold with a Sasakian 3-structure, such that \( \dim \mu^j = 1 \), for any \( x \in M \) and \( H \neq 0 \). Then \( M \) is an extrinsic sphere.

**Proof.** Let \( X \in \Gamma(TM) \), \( \forall \ X \in \Gamma(D) \). By using (1.3a) and (2.1b) we infer that

\[
g(\nabla_{\dot{X}} H, X) = g(\nabla_X J_a X, f_a X) = g(H, J_a X).
\]

On the other hand, using (1.6) and (1.8) we deduce that

\[
g(K(W_1, X) f_a Y, U) = g((\nabla_{\dot{W}_1} H) f_a Y, \nabla_{\dot{X}} H)
\]

(2.3)

The relations (2.2) and (2.3) imply

\[
g(X, Y) = g(K(W_1, X) f_a Y, X)
\]

(2.4)

But, using the symmetry properties of the tensors \( g, K \) and \( f_a \) with respect to \( g \), we get

\[
\nabla_{\dot{X}} H = 0, \ Z \in \Gamma(D)
\]

(2.5)

Now, the relations (2.5), (2.6) and Lemma 2.2 imply \( \nabla_{\dot{X}} H = 0, \ \forall \ X \in \Gamma(D) \). Taking again \( X \in \Gamma(D) \) a unit vector field and using (1.6), (1.7) and (1.8a), we deduce that

\[
\nabla_{\dot{X}} H = 0.
\]

(2.7)

Taking into account (1.3c), the fact that \( U \in \Gamma(\mu^j) \), from (2.7) and Lemma 2.2 we get

\[
\nabla_{\dot{X}} H = 0, \ \forall \ X \in \Gamma(TM) \). The proof is complete.

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Reference


5. Călin C. *Semi-invariant submanifolds of a ma


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