Local and global uniqueness theorems on differential equations of non-integer order via Bihari's and Gronwall's inequalities

S.M. Momani

Department of Mathematics and Statistics, Faculty of Science, Mutah University, P.O. Box 7. Mutah, Jordan. E-mail: smmom@mail.mutah.edu.jo / shahermm@yahoo.com

Abstract

Local and global uniqueness theorems of solutions of the non-linear differential equations

$$x^{(\alpha)}(t) = f(t, x), \alpha \in \mathbb{R}, 0 < \alpha \leq 1$$

of non-integer order have been obtained. Our method is an application of Gronwall's and Bihari's inequalities.

Key words: Fractional derivative.

Teoremas de unicidad de soluciones locales y globales de ecuaciones diferenciales no-lineales usando las desigualdades de Bihari y Gronwall

Resumen

Se obtuvieron teoremas de unicidad de soluciones locales y globales de ecuaciones diferenciales no-lineales

$$x^{(\alpha)}(t) = f(t, x), \alpha \in \mathbb{R}, 0 < \alpha \leq 1$$

de orden no-entero. Nuestro método es una aplicación de las desigualdades de Gronwall y Bihani.

Palabras clave: Derivada fraccional.

1. Introduction

Consider the initial value non-homogeneous differential equations with fractional derivative (i) subject to (ii):

(i) $x^{(\alpha)}(t) = f(t, x), \alpha \in \mathbb{R}, 0 < \alpha \leq 1,$

(ii) $x^{(\alpha-1)}(t_0) = x_0,$ (1)

where $\mathbb{R}$ is the set of real numbers, $t \in I = [0,\infty)$ and f is a real-valued function on $D = I \times \mathbb{R}^n$ into $\mathbb{R}^n$, where $\mathbb{R}^n$ denotes the real n-dimensional Euclidean space, and x_0 is a real constant.

In a recent paper, Hadid *et al* [1], used the fixed point theorem and contraction mapping principle to obtain local existence and uniqueness of solution of problem (1). Hadid [2] used Schauder's fixed-point theorem to obtain local existence, and Tychonov's fixed-point theorem to obtain global existence of solution of (1). Bassam [3] proved local existence and uniqueness theorem for (1), by using the Banach contraction mapping principle.

In this paper, we shall use Bihari's inequality [4] to obtain local uniqueness and Gronwall's inequality to obtain global uniqueness of solution of problem (1). We shall adopt the definitions and notations used in [5] and [3].

It is worth mentioning that it was shown by Hadid and Al-Shamani [6] that the solution of (1) is of the form

$$x(t) = x_0(t-t_0)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}f(s,x(s))\,ds, \quad (2)$$

where $0 < t_0 < t \le t_0 + a$, and Γ is the Gamma function, provided that the integral exists, in the Lebesgue sense.

2. Local Uniqueness

In this section, we shall prove a local uniqueness result by applying Bihari's inequality, which we state here in a suitable form.

Theorem (Bihari's inequality)

Let g be a monotone continuous function in an interval I, containing a point u_0, which vanishes nowhere in I. Let u and k be continuous functions in an interval $J = [t_0, t_0 + c]$ such that $u(J) \subset I$, and suppose that k is of fixed sign in J. Let $a \in I$. Suppose that

$$u(t) \le a + \int_{t_0}^{t} k(s)g(u(s))ds,\ t \in J.$$

Then

$$u(t) \le G^{-1}\left[G(a) + \int_{t_0}^{t} k(s)ds\right],\ t \in J,$$

where $G(u)$ is a primitive of $\frac{1}{g(x)}$, i.e. $G(u) = \int_{u_0}^{u}\frac{dx}{g(x)}$, $u \in I$.

Theorem (1): (Local uniqueness theorem)

The initial value problem (1) has a unique solution on the interval $t_0 < t \le t_0 + a$, if the function $f(t,x)$ is continuous on the region

$$0 < t_0 < t \le t_0 + a,\quad |x - x_0(t-t_0)^{\alpha-1}| \le b,$$

and such that

$$|f(t,x) - f(t,y)| \le \phi(|x-y|), \quad (3)$$

where $\phi(u)$ is a continuous non-decreasing function on $0 < u \le A$, with $\phi(0) = 0$ and

$$\int_0^A \frac{du}{\phi(u)} = +\infty. \quad (4)$$

Proof:

Assume that there exist two solutions $x(t)$ and $y(t)$ of (1), both defined in a neighbourhood at the right of t_0. We have

$$x(t) = x_0(t-t_0)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}f(s,x(s))\,ds,$$

$$y(t) = x_0(t-t_0)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}f(s,y(s))\,ds,$$

which lead easily to

$$|x(t)-y(t)| \le \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}|f(s,x(s)) - f(s,y(s))|ds.$$

It follows from (3) that

$$|x(t)-y(t)| \le \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\phi(|x(s)-y(s)|)ds,$$

$$< \varepsilon + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\phi(|x(s)-y(s)|)ds,$$

where $\varepsilon > 0$. We can now apply Bihari's inequality to obtain

$$|x(t)-y(t)| < \Phi^{-1}\left[\Phi(\varepsilon) + \frac{(t-t_0)^{\alpha}}{\alpha\Gamma(\alpha)}\right],$$

for $t \in [t_0, t_0 + a]$, (5)

where $\Phi(u)$ is a primitive of the function $\frac{1}{\phi(u)}$, and Φ^{-1} denotes the inverse of Φ.

We shall prove that the right-hand side of (5) tends toward zero as $\varepsilon \to 0$. Inasmuch as $|x(t) - y(t)|$ is independent of ε, it follows that $x(t) \equiv y(t)$, which we need. Let us remark that condition (4) implies $\Phi(\varepsilon) \to -\infty$ for $\varepsilon \to 0$, no matter how we choose the primitive of $\frac{1}{\phi(u)}$. Thus $\Phi^{-1}(u) \to 0$ as $u \to -\infty$. Consequently, when $\varepsilon \to 0$ in the inequality (5), the right-hand side tends toward zero (for all finite t).

Therefore, $x(t) = y(t)$, for $t \in [t_0, t_0 + a]$, and the theorem is proved.

Remark:

The Lipschitz condition corresponds to $\phi(u) = Lu$, for some positive constant L. Another possible choice for $\phi(u)$ is, for instance, $\phi(u) = Lu\,|\ln u|$.

3. Global Uniqueness

We shall next discuss a global uniqueness result for the initial-value problem (1) using Gronwall's inequality, which we state in the following form.

Theorem (Gronwall's inequality)

Let $a(t)$, $b(t)$, and $u(t)$ be continuous functions in $J = [t_0, t_0 + c]$, and let $b(t)$ be nonnegative in J. Suppose

$u(t) \le a(t) + \int_{t_0}^{t} b(s)u(s)ds$, $t \in J$. Then

$$u(t) \le a(t) + \int_{t_0}^{t} a(s)b(s) \exp\left[\int_{s}^{t} b(\tau)d\tau\right]ds \ , \ t \in J.$$

Theorem (2):
(Global uniqueness Theorem)

Assume that

(i) $f(t,x)$ is continuous in the region

$$D = \left\{(t, x): 0 < t_0 < t \le t_0 + a, \left\|x - x_0(t - t_0)^{\alpha-1}\right\| \le b\right\} \subset \Omega,$$

where Ω is an open (t,x)-set in $\mathbb{R}^{n+1}$.

(ii) $f(t,x)$ satisfies a local Lipschitz condition, with respect to x,

$$\|f(t, x) - f(t, y)\| \le L\|x - y\|,$$

for some positive constant L.

(iii) $x(t)$ and $\bar{x}(t)$ are solutions of (1), such that their intervals of definition have common points and $x^{(\alpha-1)}(t_0) = \bar{x}^{(\alpha-1)}(t_0)$, in such a point.
Then $x(t) = \bar{x}(t)$ on the common interval of definition.

Proof:

Assume that (t_1, t_2) is the interval where both solutions are defined. Then $t_0 \in (t_1, t_2)$. It suffices to prove that $x(t) = \bar{x}(t)$ for $t_0 \le t < t_2$.

Consider now a number T, such that $t_0 < T < t_2$. It will be fixed in the first step of the proof, but we want to point out that it can be chosen arbitrarily close to t_2. Let $K \subset \Omega$ be a compact set such that

$$(t, x(t)),\ (t, \bar{x}(t)) \in K, \quad \text{for } t \in [t_0, T].$$

The existence of the set K, with the preceding property, is the consequence of the fact that both sets $\{(t,x(t));\ t \in [t_0,T]\}$ and $\{(t,\bar{x}(t));\ t \in [t_0,T]\}$ are compact, which follows easily from the continuity of $x(t)$ and $\bar{x}(t)$.

Denote by x_0 the common value of the solutions $x(t)$ and $\bar{x}(t)$ at $t = t_0$.

For $t \in [t_0, T]$ we shall have

$$x(t) = x_0(t - t_0)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t} (t - s)^{\alpha-1} f(s, x(s))\, ds,$$

$$\bar{x}(t) = x_0(t - t_0)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t} (t - s)^{\alpha-1} f(s, \bar{x}(s))\, ds,$$

from which we get

$$\begin{aligned}\|x(t) - \bar{x}(t)\| &\le \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t} (t - s)^{\alpha-1}\|f(s, x(s)) - f(s, \bar{x}(s))\|ds, \\ &\le \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t} (t - s)^{\alpha-1}\|x(s) - \bar{x}(s)\|ds, \\ &< \varepsilon + \frac{L}{\Gamma(\alpha)}\int_{t_0}^{t} (t - s)^{\alpha-1}\|x(s) - \bar{x}(s)\|ds, \qquad (6)\end{aligned}$$

for any $\varepsilon < 0$ and $t \in [t_0, T]$.

Inequality (6) is of Gronwall type, therefore, the application of Gronwall's Theorem yields

$$\begin{aligned}\|x(t) - \bar{x}(t)\| &< \varepsilon + \frac{\varepsilon L a^{\alpha}}{\alpha\Gamma(\alpha)} \exp\left[\frac{(t - t_0)^{\alpha}}{\alpha}\right], \\ &< \varepsilon\left[1 + \frac{L a^{\alpha}}{\alpha\Gamma(\alpha)} \exp\left(\frac{(t - t_0)^{\alpha}}{\alpha}\right)\right],\end{aligned}$$

$t \in [t_0, T]$. (7)

Since ε is arbitrary, inequality (7) implies that $x(t) = \bar{x}(t)$ on $[t_0, T]$. On the other hand, T can be chosen arbitrarily close to t_2, which proves that $x(t) = \bar{x}(t)$ on $[t_0, t_2]$.

Hence the theorem is proved.

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