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Foundations of the Lie-Santilli Isotopic theory

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Abstract

Lie´s theory in its current formulation is linear, local and canonical. As such, it is inapplicable to a growing number of nonlinear, nonlocal and noncanonical systems in various fields. In this paper we review and develop a generalization of Lie's theory proposed by R.M. Santilli in 1978 then at Harvard University and today called *Lie-Santilli isotopic theory* or *Isotheory* for short. The latter theory is based on the so-called *isotopies* which are nonlinear, nonlocal and noncanonical maps of any given linear, local and canonical theory capable of reconstructing linearity, locality and canonicity in certain generalized spaces and fields. The emerging Lie-Santilli isotheory is remarkable because it presever the abstract axioms of Lie's theory while being applicable to nonlinear, nonlocal and noncanonical systems. We review the foundations of the Lie-Santilli isoalgebras and isogroups; introduce seemingly novel advances in their structure and interconnections; and show that be Lie-Santilli isotheory provides the invariance of all infinitely possible, signature-preserving, nonlinear, nonlocal and noncanonical deformations of conventional Euclidean, Minkowskian or Riemannian invariants. We finally indicate a number of applications and identify rather intriguing open mathematical problems.

Kew words: Isotopies, Lie-Santilli isoalgebras, isogroups and isorepresentations.

Fundamentos de la Teoría Isotópica de Lie-Santilli

Resumen

En su formulación actual, la teoría de Lie es lineal, local y canónica. Como tal, no es aplicable a un creciente número de sistemas no lineales, no locales y no canónicos en varios campos. En este trabajo revisamos y desarrollamos una generalización de la teoría de Lie propuesta por R.M. Santilli en 1978, quien en esos momentos se encontraba en la Universidad de Harvard. Hoy en día esta teoría se denomina *Teoría Isotópica de Lie-Santilli* o *Isoteoría*. Esta teoría está basada en las llamadas *isotopías* que son representaciones no lineales, no locales y no canónicas de una teoría lineal, local y canónica cualquiera que sea capaz de reconstruir linealidad, localidad y canonicidad en ciertos espacios y campos generalizados. La emergente Isoteoría de Lie-Santilli es notable porque preserva los axiomas abstractos de la Teoría de Lie, siendo al mismo tiempo aplicable a sistemas alineales, no locales y no canónicos. Revisamos los fundamentos de las isoálgebras e isogrupos de Lie-Santilli; introducimos avances de estructura e interconexiones aparentemente originales, y demostramos que la Isoteoría de Lie-Santilli proporciona la no varianza de todas las infinitamente posibles deformaciones -alineales, no locales, no canónicas y que preservan su configuración- de las Invariantes Euclideanas, Minkovianas o Riemanianas. Finalmente, señalamos un número de aplicaciones e identificamos algunos problemas matemáticos abiertos bastante interesantes.

Palabras clave: Isotopías, isoálgebras, isogrupos, isorrepresentaciones de Lie-Santilli.

1. Introduction

1.A. Limitations of Lie's theory. As it is well known, *Lie's theory* has permitted outstanding achievements in various disciplines. Nevertheless, in its current conception [30] and realization (see, e.g., [15]), Lie's theory is *linear, local-differential and canonical-Hamiltonian*. As such, it possesses clear limitations.

An illustration is provided by the historical distinction introduced by Lagrange [29], Hamilton [14] and others between the *exterior dynamical problems* in vacuum and the *interior dynamical problems* within physical media. Exterior problems consist of particles which can be effectively approximated as being point-like while moving within the homogeneous and isotropic vacuum under action-at-a-distance interactions (such as a space-ship in a stationary orbit around Earth). The point-like character of particles permits the validity of conventional local-differential topologies (e.g., the Zeeman topology in special relativity); the homogeneity and isotropy of space then allow the exact validity of the geometries underlying Lie's theory (such as the Riemannian geometry); and the action-at-a-distance interactions assures their representation via a potential with consequential canonical character.

Interior problems consist of extended, and therefore deformable particles moving within inhomogeneous and anisotropic physical media, with action-at-a-distance as well as contact-resistive interactions (such as a space-ship during re-entry in Earth's atmosphere). In the latter case the forces are of local-differential type (e.g., potential forces acting on the center-of-mass of the particle) as well as of nonlocal-integral type (e.g., requiring an integral over the surface of the body), thus rendering inapplicable conventional local-differential topologies; the inhomogeneity and anisotropy of the medium imply the inapplicability of conventional geometries for their quantitative treatment; while contact-resistive interactions violate Helmholtz's conditions for the existence of a potential (the *conditions of variational selfadjointness* [49]), thus implying the noncanonical character of interior systems.

We can therefore say that Lie's theory in its conventional linear, local and canonical formulation is *exactly valid* for all exterior dynamical problems, while it is *inapplicable* (and not "violated") for the more general interior dynamical problems on topological, geometrical, analytic and other grounds.

1.B. The need for a suitable generalization of Lie's theory. Lie's theory is currently applied to nonlinear, nonlocal and noncanonical systems via their simplifications into more treatable forms, e.g., via the expansion of nonlocal-integral terms into power series in the velocities and then the transformation of the system into a coordinate frame in which it admit a Hamiltonian via the Lie-Koening Theorem [49].

At times, however, nonlinear, nonlocal and nonhamiltonian systems cannot be consistently reduced or transformed into linear, local and Hamiltonian ones. An illustration exists in gravitation. The distinction between exterior and interior gravitational problems was in full use in the early part of this century (see, e.g., Schwartzschild's *two* papers, the first celebrated paper [72] on the exterior problem and the second little known paper [73] on the interior problem). The distinction was then kept in early well written treatises in the field (see, e.g., [4], [38]). The distinction was then progressively abandoned up to the current treatment of all gravitational problems, whether interior or exterior, via the same local-differential Riemannian geometry.

The above trend was based on the belief that interior dynamical problems within physical media can be effectively reduced to a collection of exterior problems in vacuum (e.g., the reduction of a space-ship during re-entry in our atmosphere to its elementary constituents moving in vacuum).

It is important for this paper to know that the above reduction is mathematically impossible. For instance, the so-called *No-Reduction Theorems* [54] prohibit the reduction of a macroscopic interior system (such as satellite during re-entry) *with a monotonically decreasing angular momentum*, to a finite collection of elementary particles each one with *a conserved angular momentum*, and viceversa.

On geometrical grounds, gravitational collapse and other interior gravitational problems are not composed of ideal points, but instead of a large number of extended and hyperdense particles (such as protons, neutrons and other particles) in conditions of total mutual penetration, as well as of compression in large numbers into small regions of space. This implies the emergence of a structure which is arbitrarily nonlinear (in coordinates and velocities), nonlocal-integral (in various quantities) and non-hamiltonian (variationally nonselfadjoint). Additional insufficiencies of the current formulation of Lie's theory and of its underlying geometries exist for the characterization of antimatter, e.g., because of the lack of a suitable (e.g., antiautomorphic) map which permits the characterization of antimatter, first, at the classical-astrophysical level, and then at the level of its elementary constituents.

Similar occurrences have recently emerged in astrophysics, superconductivity, theoretical biology and other disciplines. These occurrences establish the need for a generalization of the conventional Lie theory which is directly applicable (i.e., applicable without approximation or transformations) to nonlinear, integro-differential and variationally nonselfadjoint equations for the characterization of matter, and then possesses a suitable antiautomorphic map for the effective characterization of antimatter.

1.C: Santilli's isotopies of Lie's theory. In a seminal memoir [47] written in 1978 when at Harvard University, Santilli proposed a step-by-step generalization of the conventional formulation of Lie theory specifically conceived for nonlinear, integro-differential and noncanonical equations. The generalized theory was subsequently studied by Santilli in ref.s ([48]-[71]), as well as by a number of mathematicians and theoreticians, and it is today called *Lie-Santilli isotopic theory* or *isotheory* (see papers [1], [2], [8], [11], [12], [16]-[23], [25], [32], [33], [35]-[37], [40]-[43], monographs [3], [24], [31], [74] and additional references quoted therein).

A main characteristic of the Lie-Santilli isotheory, which distinguishes it from all other possible generalizations, is its "isotopic" character intended (from the Greek meaning of the word) as the capability of preserving the original Lie axioms. More specifically, Santilli's isotopies are maps of any given linear, local and canonical structure into its most general possible nonlinear, nonlocal and noncanonical forms which are capable of reconstructing linearity, locality and canonicity in certain generalized isospaces and isofields within a fixed system of local coordinates.

The latter property is remarkable, mathematically and physically, inasmuch as it permits the preservation of the abstract Lie theory and the transition from exterior to interior problems via a more general *realization* of the same theory.

Another main characteristic of the Lie-Santilli isotheory is that of admitting a novel antiautomorphic map, called *isoduality*, which has resulted to be effective for the characterization of antimatter at the classical as well as operator levels.

It should be indicated that Santilli [47] submitted his isotopic theory *as a particular case* of a yet more general theory today called *Santilli's Lie-admissible theory* or *Lie-Santilli genotopic theory* where the term *genotopic* is used (in its Greek meaning) to "induce configuration", and interpreted in the sense of violating the original Lie axioms, but inducing covering Lie-admissible axioms.

This paper is written by a theoretical physicist for mathematicians and it is solely devoted to the Lie-Santilli isotheory. A study of the broader Lie-Santilli genotheory is contemplated as a future work. In Sect. 2 we outline the methodological foundations of the theory. The isotopies of Lie's theory are presented in Sect. 3 jointly with new developments, such as a study of the transition from the Lie-Santilli isogroups to the corresponding isoalgebras. As an illustration of the capabilities of the isotheory, we prove its "direct universality" in gravitation, that is, the achievement of the symmetries of all possible gravitational metrics (universality), directly in the frame of the experimenter (direct universality). A number of fundamental open mathematical problems will be identified during the course of our analysis.

A comprehensive mathematical presentation of the Lie-Santilli isotheory up to 1992 is available in monograph [74]. A historical perspective is available in monograph [31]. Recent mathematical studies on isomanifolds (today called *Tsagas isomanifolds*) have been conducted in ref. [75] which is a topological complement of the algebraic studies of this paper.

2. Isotopies and Isodualities of Contemporary Mathematical Structures

2.A: Statement of the problem. Lie's theory is the embodiment of the virtual entirety of contemporary mathematics by encompassing: the theory of numbers; differential and exterior calculus; vector and metric spaces; geometry, algebra and topology; functional analysis; and others. Santilli's isotopies of Lie's theory require the isotopic lifting of all these mathematical methods.

The most recent isotopies of contemporary mathematical methods has been published in this Journal in three preceding papers by Santilli [71]. To avoid un-necessary repetition, we shall herein assume the entirety of the content of these papers and refer to them as I, II and III (e.g., Sect. I.3 or Eq. (III.3.33). Additional studies via a different type of isotopies are available in monographs [61] together with numerous applications. In this section we shall mainly recall the fundamental notions, and refer to papers I, II and III for all details.

2.B. Isotopies and isodualities of the unit. The fundamental isotopies from which all others can be uniquely derived are given by the liftings of the n-dimensional unit I = diag. (1, 1, ..., 1) of the current formulation of Lie's theory into a matrix $\hat{1}$ of the same dimension of I, but with unrestricted functional dependence of its elements in the local coordinates x, their derivatives with respect to an independent variable of arbitrary order, $\dot{x}$, $\ddot{x}$, ... as well as any needed additional quantity [47], [49b], [61a], [1-71],

$$I \rightarrow \hat{1} = \hat{1}(x, \dot{x}, \ddot{x}, ...) . \tag{2.1}$$

The *isotopies* occur when $\hat{1}$ preserves all the topological characteristics of I, such as nowhere-degeneracy, real-valuedness and positive-definiteness.

Once the unit is generalized, there is the natural emergence of the map [52], [53], [61a], [1-71],

$$\hat{1} \rightarrow \hat{1}^d = -\hat{1}, \tag{2.2}$$

called by Santilli *isoduality* which provides an antiautomorphic image of all formulations based on $\hat{1}$.

The above liftings were classified by the author [22] into:

Class I (generalized units that are sufficiently smooth, bounded, nowhere degenerate, Hermitean and positive-definite, characterizing the *isotopies* properly speaking);

Class II (the same as Class I although $\hat{1}$ is negative-definite, characterizing *isodualities*);

Class III (the union of Class I and II);

Class IV (Class III plus singular isounits); and

Class V (Class IV plus unrestricted generalized units, e.g., realized via discontinuous functions, distributions, lattices, etc.).

All isotopic structures studied in this paper also admit the same classification which will be omitted for brevity. Hereon we shall generally study isotopies of Classes I and II, at times treated in a unified way via those of Class III whenever no ambiguity arises. Santilli's isotopies of Classes IV and V are vastly unexplored at this writing.

2.C. Isotopies and isodualities of contemporary mathematics. Lie's theory is constructed over ordinary fields F(a,+,×) hereon assumed to be of characteristic zero (the fields of real $\Re$, complex C and quaternionic numbers Q) with generic elements a, addition $a_1 + a_2$, multiplication $a_1 a_2 := a_1 \times a_2$, additive unit 0, a + 0 = 0 + a ≡ a, and multiplicative unit I, a×I = I×a ≡ a, ∀ a, a_1, a_2 ∈ F.

The Lie-Santilli isotheory is based on a generalization of the very notion of numbers and, consequently of fields (see ref. [39], comprehensive mathematical studies [59] and monographs [61] for applications).

Consider a Class I lifting of the unit I of F, I → $\hat{1}$ with $\hat{1}$ being *outside* the original set, $\hat{1}$ ∉ F. In order for $\hat{1}$ to be the left and right unit of the new theory, it is necessary to lift the conventional associative multiplication ab into the so-called *isomultiplication* [47]

$$a\, b := a \times b \Rightarrow a * b := a \times T \times a = a\, T\, b, \tag{2.3}$$

where the quantity T is fixed and called the *isotopic element*. Whenever $\hat{1} = T^{-1}$, $\hat{1}$ is the correct left and right unit of the theory, $\hat{1} * a = a * \hat{1} = a$, $\forall a \in F$, in which case (only) $\hat{1}$ is called the *isounit*. In turn, the liftings $I \to \hat{1}$ and $\times \to *$, imply the generalization of fields into the Class I structure

$$\hat{F}_I = \{ (\hat{a},+,*) \mid \hat{a} = a\hat{1};\ * = \times T \times; \hat{1} = T^{-1} \}, \qquad (2.4)$$

called *isofields*, with elements $\hat{a} \in \hat{F}$ called *isonumbers* [59].

All conventional operations among numbers are evidently generalized in the transition from numbers to isonumbers. In fact, we have: $a + b \to \hat{a} + \hat{b} = (a + b)\hat{1}$; $a_1 \times a_2 \to \hat{a}_1 * \hat{a}_2 = \hat{a}_1 T \hat{a}_2 = (a_1 a_2)\hat{1}$; $a^{-1} \to \hat{a}^{\hat{1}} = a^{-1}\hat{1}$; $a / b = c \to \hat{a} \hat{/} \hat{b} = \hat{c}$, $a^{\frac{1}{2}} \to \hat{a}^{\hat{\frac{1}{2}}} = a^{\frac{1}{2}}\hat{1}^{\frac{1}{2}}$; etc. Thus, conventional squares $a^2 = aa$ have no meaning under isotopy and must be lifted into the *isosquare* $\hat{a}^{\hat{2}} = \hat{a} * \hat{a}$. The *isonorm* is

$$\hat{|}\, \hat{a}\, \hat{|} = (\bar{a}\, a)^{1/2}\hat{1} = |a|\hat{1} \in \hat{F}, \qquad (2.5)$$

where $\bar{a}$ denote the conventional conjugation in F and $|a|$ the conventional norm. Note that the *isonorm is positive-definite* (for isofields of Class I), as a necessary condition for isotopies.

The isotopic character of the lifting $I \to \hat{1}$ is confirmed by the fact that the isounit $\hat{1}$ verifies all axioms of I, $\hat{1} * \hat{1} * \ldots * \hat{1} = \hat{1}$, $\hat{1} \hat{/} \hat{1} = \hat{1}$, $\hat{1}^{\hat{\frac{1}{2}}} = \hat{1}$, etc.

The *isodual isofields* are the antihomomorphic image of $\hat{F}(\hat{a},+,*)$ induced by the map $\hat{1} \to \hat{1}^d = -\hat{1}$ and are given by the Class II structures

$$\hat{F}^d_{II} = \{ (\hat{a}^d,+,*^d) \mid \hat{a}^d = \bar{a}\hat{1}^d;\ *^d = \times T^d \times,\ T^d = -T,\ \hat{1}^d = -\hat{1} \}, \qquad (2.6)$$

in which the elements $\hat{a}^d = \bar{a}\hat{1}^d$ are called *isodual isonumbers*. For real numbers we have $n^d = -n$, for complex numbers we have $c^d = -\bar{c}$, where $\bar{c}$ is the ordinary complex conjugate, and for quaternions in matrix representation we have $q^d = -q^\dagger$, where $\dagger$ is the Hermitean conjugate.

It is to be observed that the imaginary number i is *isoselfdual*, i.e., invariant under isoduality, $i^d = -\bar{i} = i$, and the conjugation of a complex number is given by $(n + i \times m)^d = n^d + i^d \times^d m^d = -n + (-\bar{i})(-\times)(-m) = -n + im$.

The *isodual isosum* is given by $\hat{a}^d + \hat{b}^d = (\bar{a} + \bar{b})\hat{1}^d$, while for the *isodual isomultiplication* we have $\hat{a}^d *^d \hat{b}^d = \hat{a}^d T^d \hat{b}^d = -\hat{a}^d T \hat{b}^d = (\bar{a}\,\bar{b})\hat{1}^d$.

An important property is that *the norm of isodual isofields is negative-definite*,

$$\hat{|}\hat{a}^d\hat{|}^d = |\bar{a}|\hat{1}^d = -\hat{|}\,\hat{a}\,\hat{|}. \qquad (2.7)$$

The latter property has nontrivial implications. For instance, it implies that *physical quantities defined on an isodual isofield, such as time, energy, angular momentum, etc., are negative-definite*. For these reasons, isodual theories provide a novel and intriguing characterization of antimatter [61].

Note also that, *as a necessary condition for isotopies (isodualities) all isofields $\hat{F}_I(\hat{a},+,*)$ (isodual isofields $\hat{F}_{II}{}^d(\hat{a}^d,+,*^d)$) are isomorphic (antiisomorphic) to the original field* $F(a,+,*)$. The reader should be aware that the distinction between real, complex and quaternionic numbers is lost under isotopies because all possible numbers are unified by the isoreals owing to the freedom in the generalized unit [26].

As an illustrative example, the isounit used by Animalu [1] for the representation of the Cooper pair in superconductivity is given by

$$\hat{1} = I\, e^{tN\int d^3x\, \psi^\dagger_\uparrow(r)\, \phi_\downarrow(r)}, \qquad (2.8)$$

where t represents time, N is a positive real constant, and $\psi_\uparrow$ and $\bar{\phi}_\downarrow$ are the wavefunctions of the two electrons of the Cooper pair with related orientation of their spin. *Animalu's isounit* (2.8) therefore represents the *nonlocal-integral* contributions due to the wave overlapping of the two electrons in the Cooper pairs.

We also recall the still more general *genofields* [59], characterized first by an isotopy of conventional fields, and then by the differentiation of the isomultiplications to the right $\hat{a}>\hat{b} = \hat{A}\times R\times\hat{b}$ from that to the left $\hat{a}<\hat{b} = \hat{a}\times S\times\hat{b}$, $\hat{a}>\hat{b} \neq \hat{a}<\hat{b}$, $R \neq S$. The important property is that all abstract axioms of a field are verified per *each* ordered isomultiplication thus yielding one *genofield* $\hat{F}^>(\hat{a},+,>)$ for the multiplication to the right and a different one $^<\hat{F}(\hat{a},+,<)$ for the multiplication to the left. The latter genofields are at the foundation of the *Lie-Santilli genotopic theory* or *genotheory* for short with a Lie-admissible (rather than Lie-isotopic) structure.

A still more general formu.lation is currently under study via the *hyperstructures* (see, e.g., monograph [7]). In essence, the genotopic elements R and S are irreducible and fixed in the genotopic products $\hat{a}>\hat{b}$ and $\hat{a}<\hat{b}$. In the transition to the hyperstructure, the genotopic element R and S assume finite or infinite and ordered or non-ordered sets of values.

We finally recall the liftings characterized by the generalization of the sum + and related additive unit 0, e.g., $+ \rightarrow \hat{+} = + K+$, $0 = K \neq 0$, $K \in F$ $(a \hat{+} b = a + K + b)$ called *pseudo isotopies* [59], which *do not* preserve the axioms of a field (e.g., closure under the distributive law is not verified under the conventional × or isotopic * multiplication and the addition $\hat{+}$). Thus, *pseudoisofields are not fields*. For these and other reasons (e.g., the general divergence of the exponentiation), physical applications are restricted to iso- and geno-fields, while the pseudoiso- and pseudogeno-fields have a mere analytical interest at this writing.

Despite the above advances, studies on the isonumber theory need further investigations. To begin, the entire conventional number theory (including all familiar theorems on factorization etc.) can be subjected to an isotopic lifting of Class I. Moreover, we have the birth of new numbers without counterpart in the current number theory, such as the isonumbers of Class II (with negative-definite unit), of Class IV (with singular isounits) and Class V (with distributions or discontinuous functions as isounits). All the above liftings then admit a further enlargement via the differentiation of the multiplications to the right and to the left, and then yet more general formulations via the multivalued hyperstructures.

The isotopies and isodualities of fields outlined above admit corresponding lifting of all conventional mathematical quantities defiend on them, such as vector and metric spaces, functional analysis, etc. for which we refer for brevity to [61], [71].

One can begin to understand the vastity of the Lie-Santilli isotheory as compared to the conventional formulation of Lie's theory by noting that the above hierarchy of fields implies a corresponding hierarchy of Lie-isotopic theories, which includes a corresponding hietarchy of isospaces, isoalgebras, isogroups, etc.

3. Isotopies and Isodualities of Enveloping Algebras, Lie Algebras, Lie Groups, Symmetries, Representation Theory and Their Applications

As recalled in Sect. 1, Lie's theory (see [13], [15] and [76]) is centrally dependent on the basic n-dimensional unit $I = \text{diag.}(1, 1, ..., 1)$ in all its major branches, such as enveloping algebras, Lie algebras, Lie groups, representation theory, etc. The main idea of the Lie-Santilli theory [47], [49], [61], [62] is the reformulation of the entire conventional theory with respect to the most general possible, integro-differential isounit $\hat{1}(x, \dot{x}, \ddot{x}, ...)$.

One can therefore see from the very outset the richness and novelty of the isotopic theory. In fact, it can be classified into five main classes as occurring for isofields, isospaces, etc., and admits novel realizations and applications, e.g., in the construction of the symmetries of deformed line elements of metric spaces.

3.A. Isotopies and isodualities of universal enveloping associative algebras. Let ξ be a universal enveloping associative algebra [15] over a field F (of characteristic zero) with generic elements A, B, C,... , trivial associative product AB and unit I. Their isotopes $\hat{\xi}$ were first introduced in [47] under the name of *isoassociative envelopes*. They coincide with ξ as vector spaces but are equipped with the isoproduct so as to admit $\hat{1}$ as the correct (right and left) unit

$$A * B = ATB, \; \hat{1} * A = A * \hat{1} \equiv A \quad \forall A \; \hat{\xi}, \quad \hat{1} = T^{-1}. \tag{3.1}$$

Let $\xi = \xi(L)$ be the universal enveloping algebra of an N-dimensional Lie algebra L with ordered basis $\{X_k\}$, $k = 1, 2, ..., N$, $[\xi(L)]^{-} \approx L$ over F, and let the infinite-dimensional basis of $\xi(L)$ be given by the Poincaré-Birkhoff-Witt theorem [15]. A fundamental result due to Santilli ([47], [59], Vol. II, p. 154-163) is as follows

Theorem 3.1. *The cosets of $\hat{1}$ and the standard, isotopically mapped monomials*

$\hat{1}$, X_k $\quad X_i * X_j \ (i \leq j)$, $\quad X_i * X_j * X_k \ (i \leq j \leq k)$,

. . . (3.2)

form a basis of the universal enveloping isoassociative algebra $\hat{\xi}(L)$ *of a Lie algebra* L.

A first important consequence is that the isotopies of conventional exponentiation are given by the expression, called *isoexponentiation*, for $\hat{w} \in \hat{F}$,

$$e_{\hat{\xi}}^{i\hat{w}*X} = \hat{1} + (i\hat{w}*X)/1! + (i\hat{w}*X)*(i\hat{w}*X)/2! + ... =$$

$$= \hat{1}\{e^{iwTX}\} = \{e^{iXTw}\}\hat{1}. \qquad (3.3)$$

The implications of Theorem 3.1 also emerge at the level of functional analysis because all structures defined via the conventional exponentiation must be suitably lifted into a form compatible with Theorem 3.1. As an example, Fourier transforms are structurally dependent on the conventional exponentiation. As a result, they must be lifted under isotopies into the expressions [23]

$$f(x) = (1/2\pi)\int_{-\infty}^{+\infty} g(k) * e_{\hat{\xi}}^{ikx}\, dk\,,$$

$$g(k) = (1/2\pi)\int_{-\infty}^{+\infty} f(x) * e_{\hat{\xi}}^{-ikx}\, dx, \qquad (3.4)$$

with similar liftings for Laplace transforms, Dirac-delta distribution, etc., not reviewed here for brevity.

On physical grounds, Theorem 3.1 implies that the isotransform of a gaussian in isofunctional analysis is given by [23]

$$f(x) = N * e_{\hat{\xi}}^{-x/2a^2} = Ne^{-x^2T/2a^2} \rightarrow$$

$$\rightarrow g(k) = N * e_{\hat{\xi}}^{-k^2a^2/2} = N'e^{-k^2Ta^2/2} \qquad (3.5)$$

As a result, the widths are of the type $\Delta x \approx aT^{-\frac{1}{2}}$, $\Delta k \approx a^{-1}T^{-\frac{1}{2}}$. It then follows that the isotopies imply the loss of the conventional uncertainties $\Delta x\, \Delta k \approx 1$ in favor of the local *isouncertainties* [61b]

$$\Delta x\, \Delta k \approx \hat{1}, \qquad (3.6)$$

which illustrate the nontriviality of the the isotopy.

The *isodual isoenvelopes* $\hat{\xi}^d$ are characterized by the isodual basis $X_k{}^d = -X_k$ defined with respect to the isodual isounits $\hat{1}^d = -\hat{1}$ and isodual isotopic element $T^d = -T$ over the isodual isofields $\hat{F}^d$. The *isodual isoexponentiation* is then given by

$$e_{\hat{\xi}^d}^{i^d w^d \times^d X^d} = \hat{1}^d\{e^{iwTX}\} = -e_{\hat{\xi}}^{iwX} \qquad (3.7)$$

and plays an important role for the characterization of antiparticles as possessing negative-definite energy and moving backward in time (as necessary when using isodual isofields).

It is easy to see that Theorem 3.1 holds, as originally formulated [47], for envelopes now called of Class III, thus unifying isoenvelopes $\hat{\xi}$ and their isoduals $\hat{\xi}^d$. In fact, the theorem was conceived to unify with one single Lie algebra basis X_k nonisomorphic compact and noncompact algebras of the same dimension N (see the example of Section 3.E).

The isotopy $\xi \rightarrow \hat{\xi}$ is not a conventional map because the local coordinates x, the infinitesimal generators X_k and the parameters w_k are not changed by assumption, while the underlying unit and rtelated associative product are changed. Also, in the operator realization the Lie and Lie-Santilli isotheory can be linked by nonunitary transformations $UU^\dagger = \hat{1} \neq I$, for which

$$I \rightarrow \hat{1} = U\, I\, U^\dagger,\ A\, B \rightarrow$$

$$\rightarrow U\, A\, B\, U^\dagger = A' * B' = A'\, T\, B',\ T = (U\, U^\dagger)^{-1}. \qquad (3.8)$$

where $A' = UAU^\dagger$, $B' = UBU^\dagger$. The lack of equivalence of the two theories is further illustrated by the inequivalence between conventional eigenvalue equations,

$$H|b\rangle = E|b\rangle, \quad H = H^\dagger, \quad E \in \Re(n,+,\times),$$

and their isotopic form in the same Hamiltonian [11-71]

$$H * |\hat{b}\rangle = H\,T\,|\hat{b}\rangle = \hat{E} * |\hat{b}\rangle \equiv E'|\hat{b}\rangle, \quad H = H^\dagger,$$

where $E' \in \hat{\Re}(\hat{n},+,\times)$, with consequential *different eigenvalues for the same operator* H, $E' \neq E$ (see Section 3.E for an example). We therefore expect the weights of the Lie and Li-Santilli theories to be different.

3.B. Isotopies and isodualities of Lie algebras. A (finite-dimensional) isospace $\hat{L}$ over the isofield $\hat{F}$ of isoreal $\hat{\Re}(\hat{n},+,*)$ or isocomplex numbers with isotopic element T and isounit $\hat{1} = T^{-1}$ is called a *Lie-Santilli algebra* over $\hat{F}$ (see the original contributions [47], [49], [61], [62], independent studies [3], [24], [31], [74] and references quoted ytherein), sametimes called *isoalgebra* (when no confusion with the isotopies of non-Lie algebras arises), when there is a composition $[A\hat{,}B]$ in $\hat{L}$, called *isocommutator*, which is isolinear (i.e., satisfies condition (2.40)) and such that for all A, B, C $\in \hat{L}$

$$[A\hat{,}B] = -[B\hat{,}A], \tag{3.9a}$$

$$[A\hat{,}[B\hat{,}C]] + [B\hat{,}[C\hat{,}A]] + [C\hat{,}[A\hat{,}B]] = 0, \tag{3.9b}$$

$$[A * B\hat{,}C] = A * [B\hat{,}C] + [A\hat{,}C] * B. \tag{3.9c}$$

The isoalgebras are said to be: *isoreal (isocomplex)* when $\hat{F} = \hat{\Re}$ ($\hat{F} = \hat{C}$), and *isoabelian* when $[A\hat{,}B] = 0$ $\forall$ A, B $\in \hat{L}$. A subset $\hat{L}_0$ of $\hat{L}$ is said to be an *isosubalgebra* of $\hat{L}$ when $[\hat{L}_0\hat{,}\hat{L}_0] \subset \hat{L}_0$ and an *isoideal* when $[\hat{L}\hat{,}\hat{L}_0] \subset \hat{L}_0$. A maximal isoideal which verifies the property $[\hat{L}\hat{,}\hat{L}_0] = 0$ is called the *isocenter* of $\hat{L}$. For the isotopies of conventional notions, theorems and properties of Lie algebras see [74].

We recall the *isotopic generalizations of the celebrated Lie's First, Second and Third Theorems* introduced in ref. [47], but which we do not review here for brevit (see [49b], [61b], [74]). For instance, the isotopic second theorem reads

$$[X_i\hat{,}X_j] = X_i * X_j - X_j * X_i =$$

$$X_i\, T(x,\ldots)\, X_j - X_j\, T(x,\ldots)\, X_i = \hat{C}_{ij}^{k}(x,\dot{x},\ddot{x},\ldots) * X_k, \tag{3.10}$$

where the $\hat{C}$'s are called the *structure functions*, generally have an explicit dependence on the underlying isospace (see the example of Section 3.E), and verify certain restrictions from the Isotopic Third Theorem.

Let L be an N-dimensional Lie algebra with conventional commutation rules and structure constants C_{ij}^{k} on a space S(x,F) with local coordinates x over a field F, and let L be (homomorphic to) the antisymmetric algebra $[\xi(L)]^-$ attached to the associative envelope $\xi(L)$. Then $\hat{L}$ can be equivalently defined as (homomorphic to) the antisymmetric algebra $[\hat{\xi}(L)]^-$ attached to the isoassociative envelope $\hat{\xi}(L)$ ([47], [49], [74]). In this way, an infinite number of isoalgebras $\hat{L}$, depending on all possible isounits $\hat{1}$, can be constructed via the isotopies of one single Lie algebra L. It is easy to prove the following result:

Theorem 3.2. *The isotopies* $L \rightarrow \hat{L}$ *of an N-dimensional Lie algebra* L *preserve the original dimensionality.*

In fact, the basis e_k, k = 1, 2, ..., N of a Lie algebra L is not changed under isotopy, except for renormalization factors denoted $\hat{e}_k$. Let the commutation rules of L be given by $[e_i, e_j] = C_{ij}^{k}\, e_k$.

The isocommutation rules of the isotopes $\hat{L}$ are

$$[\hat{e}_i\hat{,}\hat{e}_j] = \hat{e}_i\, T\, \hat{e}_j - \hat{e}_j\, T\, \hat{e}_i = \hat{C}_{ij}^{k}(x,\dot{x},\ddot{x},\ldots)\, \hat{e}_k, \tag{3.11}$$

where $\hat{C} = CT$. One can then see in this way the necessity of lifting the structure <constants> into structure <functions>, as correctly predicted by the Isotopic Second Theorem.

The structure theory of the above isoalgebras is still unexplored to a considerable extent. In the following we shall show that the main lines of the conventional structure of Lie theory do indeed admit a consistent isotopic lifting. To begin, we here introduce the *general isolinear and isocomplex Lie-Santilli algebras* denoted $G\hat{L}(n,\hat{C})$ as the vector isospaces of all n×n complex matrices over $\hat{C}$. It is easy to see that they are closed under isocommutators as in the conventional case. The *isocenter* of $G\hat{L}(n,\hat{C})$ is then given by $\hat{a} * \hat{1}$, $\forall$ $\hat{a} \in \hat{\Re}$. The subset of all complex n×n matrices with null trace is also closed under isocommutators. We shall call it the *special, complex, isolinear isoalgebra* and denote it with $S\hat{L}(n,\hat{C})$. The subset of all antisymmetric n×n real matrices X, $X^t = -X$, is also closed under isocommutators, it is called the *isoorthogonal algebra*, and it is denoted with $\hat{O}(n)$.

By proceeding along similar lines, we classify all classical, non-exceptional, Lie-Santilli algebras over an isofield of characteristic zero into the isotopes of the conventional forms, denoted with $\hat{A}_n$, $\hat{B}_n$, $\hat{C}_n$ and $\hat{D}_n$ each one admitting realizations of Classes I, II, III, IV and V (of which only Classes I, II and III are studied herein). In fact, $\hat{A}_{n-1} = S\hat{L}(n,\hat{C})$; $\hat{B}_n = \hat{O}(2n+1, \hat{C})$; $\hat{C}_n = S\hat{P}(n,\hat{C})$; and $\hat{D}_n = \hat{O}(2n,\hat{C})$. One can begin to see in this way the richness of the isotopic theory as compared to the conventional theory.

The notions of *homomorphism, automorphism and isomorphism* of two isoalgebras $\hat{L}$ and $\hat{L}'$, as well as of *simplicity* and *semisimplicity* are the conventional ones. Similarly, all properties of Lie algebras based on the addition, such as the *direct and semidirect sums*, carry over to the isotopic context unchanged (because of the preservation of the conventional additive unit 0).

An *isoderivation* $\hat{D}$ of an isoalgebra $\hat{L}$ is an isolinear mapping of $\hat{L}$ into itself satisfying the property

$$\hat{D}([A, B]) = [\hat{D}(A) \hat{,} B] + [A \hat{,} \hat{D}(B)] \quad \forall A, B \in \hat{L}. \tag{3.12}$$

If two maps $\hat{D}_1$ and $\hat{D}_2$ are isoderivations, then $\hat{a} * \hat{D}_1 + \hat{b} * \hat{D}_2$ is also an isoderivation, and the isocommutators of $\hat{D}_1$ and $\hat{D}_2$ is also an isoderivation. Thus, the set of all isoderivations forms a Lie-Santilli algebra as in the conventional case.

The isolinear map $a\hat{d}(\hat{L})$ of $\hat{L}$ into itself defined by

$$a\hat{d}\, A(B) = [A \hat{,} B], \quad \forall A, B \in \hat{L}, \tag{3.13}$$

is called the *isoadjoint map*. It is an isoderivation, as one can prove via the iso-Jacobi identity. The set of all $a\hat{d}(A)$ is therefore an isolinear isoalgebra, called *isoadjoint algebra* and denoted $\hat{L}_a$. It also results to be an isoideal of the algebra of all isoderivations as in the conventional case.

Let $\hat{L}^{(o)} = \hat{L}$. Then $\hat{L}^{(1)} = [\hat{L}^{(o)} \hat{,} \hat{L}^{(o)}]$, $\hat{L}^{(2)} = [\hat{L}^{(1)}, \hat{L}^{(1)}]$, etc., are also isoideals of $\hat{L}$. $\hat{L}$ is then called *isosolvable* if, for some positive integer n, $\hat{L}^{(n)} \equiv 0$. Consider also the sequence $L_{(o)} = L$, $\hat{L}_{(1)} = [\hat{L}_{(o)} \hat{,} \hat{L}]$, $\hat{L}_{(2)} = [\hat{L}_{(1)} \hat{,} \hat{L}]$, etc. Then $\hat{L}$ is said to be *isonilpotent* if, for some positive integer n, $\hat{L}_{(n)} \equiv 0$. One can then see that, as in the conventional case, an isonilpotent algebra is also isosolvable, but the converse is not necessarily true.

Let the *isotrace* of a matrix be given by the element of the isofield [61]

$$\hat{Tr}\, A = (Tr\, A)\hat{1} \in \hat{F}, \tag{3.14}$$

where Tr A is the conventional trace. Then $\hat{Tr}(A * B) = (\hat{Tr}\, A) * (\hat{Tr}\, B)$ and $\hat{Tr}(B\, A\, B^{-1}) = \hat{Tr}\, A$. Thus, the $\hat{Tr}\, A$ preserves the axioms of Tr A, by therefore being a correct isotopy. Then the isoscalar product

$$(A \hat{,} B) = \hat{Tr}[(A\hat{d}X) * (A\hat{d}\, B)], \tag{3.15}$$

is here called the *isokilling form*. It is easy to see that $(A \hat{,} B)$ is symmetric, bilinear, and verifies the property $(A\hat{d}X(Y) \hat{,} Z) + (Y \hat{,} Ad\, X(Z)) = 0$, thus being a correct, axiom-preserving isotopy of the conventional Killing form.

Let e_k, k = 1, 2, ..., N, be the basis of L with one-to-one invertible map $e_k \to \hat{e}_k$ to the basis of $\hat{L}$. Generic elements in $\hat{L}$ can then be written in terms of local coordinates x, y, z, $A = x^i\hat{e}_i$ and $B = y^j\hat{e}_j$, and

$$C = z^k \hat{e}_k = [A \hat{,} B] = x^i\, y^j\, [\hat{e}_i \hat{,} \hat{e}_j] = x^i\, x^j\, \hat{C}_{ij}{}^k\, \hat{e}_k\ . \tag{3.16}$$

Thus,

$$[A\hat{d}\, A(B)]^k = [A \hat{,} B]^k = \hat{C}_{ij}{}^k\, x^i\, x^j. \tag{3.17}$$

We now introduce the *isocartan tensor* $\tilde{g}_{ij}$ of an isoalgebra $\hat{L}$ via the definition
$(A \hat{,} B) = \tilde{g}_{ij}\, x^i\, y^j$ yielding

$$\tilde{g}_{ij}(x, \dot{x}, \ddot{x}, ...) = \hat{C}_{ip}{}^k\, \hat{C}_{jk}{}^p. \tag{3.18}$$

Note that the isocartan tensor has the general dependence of the isometric tensor of Section 2.C, thus confirming the inner consistency among the various branches of the isotopic theory. In particular, the isocartan tensor is generally *nonlinear, nonlocal and noncanonical* in all variables x, $\dot{x}$, $\ddot{x}$, ... This clarifies that isotopic generalization of the Riemannian spaces studied in tref. [60] $R(x,g,\Re) \Rightarrow \hat{R}(x,\hat{g},\hat{\Re})$, $\hat{g} = \hat{g}(s, x, \dot{x}, \ddot{x}, ...)$, has its origin in the very structure of the Lie-isotopic theory.

The isocartan tensor also clarifies another fundamental point of Section 1, that the isotopies naturally lead to an arbitrary dependence in the velocities and accelerations, exactly as needed for realistic treatment of the problems identified in Section 1, and that their restriction to the nonlinear dependence on the coordinates x only, as generally needed for the exterior (e.g., gravitational) problem, would be manifestly un-necessary.

The isotopies of the remaining aspects of the structure theory of Lie algebras can be completed by the interested reader. Here we limit ourselves to recall that when the isocartan form is positive- (or negative-) definite, $\hat{L}$ is compact, otherwise it is noncompact. Then it is easy to prove the following

Theorem 3.3. *The Class III liftings $\hat{L}$ of a compact (noncompact) Lie algebra L are not necessarily compact (noncompact).*

The identification of the remaining properties which are not preserved under liftings of Class III is an instructive task for the interested reader. For instance, if the original structure is irreducible, its isotopic image is not necessarily so even for Class I, trivially, because the isounit itself can be reducible, thus yielding a reducible isotopic structure.

Let $\hat{L}$ be an isoalgebra with generators X_k and isounit $\hat{1} = T^{-1} > 0$. From Equations (3.7) we then see that the *isodual Lie-Santilli algebras* $\hat{L}^d$ of $\hat{L}$ is characterized by the isocommutators

$$[X_i \hat{,} X_j]^d = -[X_i \hat{,} X_j] = \hat{C}_{ij}{}^{k(d)} X^d{}_k , \quad \hat{C}_{ij}{}^{k(d)} = -\hat{C}_{ij}{}^k . \tag{3.19}$$

$\hat{L}$ and $\hat{L}^d$ are then (anti) isomorphic. Note that the isoalgebras of Class III contain all Class I isoalgebras $\hat{L}$ and all their isoduals $\hat{L}^d$. The above remarks therefore show that the Lie-Santilli theory can be naturally formulated for Class III, as implicitly done in the original proposal [47]. The formulation of the same theory for Class IV or V is however considerably involved on technical grounds thus requiring specific studies.

The notion of isoduality applies also to conventional Lie algebras L, by permitting the identification of the *isodual Lie algebras* L^d via the rule [52], [53]

$$[X_i , X_j]^d = X^d{}_i I^d X^d{}_j - X^d{}_j I^d X^d{}_i = -[X_i , X_j] =$$

$$= C_{ij}{}^{k(d)} X^d{}_k , \quad C_{ij}{}^{k(d)} = -C_{ij}{}^k . \tag{3.20}$$

Note the necessity of the isotopies for the very construction of the isodual of conventional Lie algebras. In fact, they require the nontrivial lift of the unit $I \Rightarrow I^d = (-I)$, with consequential necessary generalization of the Lie product AB − BA into the isotopic form ATB − BTA.

For realizations of the Lie-Santilli isoalgebras in classical and operator mechanics, we refer the reader for brevity to ref.s [61], [11-71].

3.C. Isotopies and isodualities of Lie groups. A *right Lie-Santilli group* $\hat{G}$ (see the original contributions [47], [49], [61], [62], inependent monographs [3], [24], [31], [74] and papers quoted therein) on an isospace $\hat{S}(x,\hat{F})$ over an isofield $\hat{F}$, $\hat{1} = T^{-1}$ (of isoreal $\hat{\mathfrak{R}}$ or isocomplex numbers $\hat{C}$), also called *isotransformation group* or *isogroup*, is a group which maps each element $x \in \hat{S}(x,\hat{F})$ into a new element $x' \in \hat{S}(x,\hat{F})$ via the isotransformations $x' = \hat{U} * x = \hat{U} T x$, T fixed, such that: (1) The map $(U, x) \rightarrow \hat{U} * x$ of $\hat{G} \times \hat{S}(x,\hat{F})$ onto $\hat{S}(x,\hat{F})$ is isodifferentiable; (2) $\hat{1} * \hat{U} = \hat{U} * \hat{1} = \hat{U}\ \forall\ \hat{U} \in \hat{G}$; and (3) $\hat{U}_1 * (\hat{U}_2 * x) = (\hat{U}_1 * \hat{U}_2) * x,\ \forall\ x \in \hat{S}(x,\hat{F})$ and $\hat{U}_1 , \hat{U}_2 \in \hat{G}$. A *left isotransformation group* is defined accordingly.

The notions of *connected or simply connected transformation groups* carry over to the isogroups in their entirety. We consider hereon the connected isotransformation groups. Right or left isogroups are characterized by the following laws [47]

$$\hat{U}(0) = \hat{1}, \quad \hat{U}(\hat{w}) * \hat{U}(\hat{w}') = \hat{U}(\hat{w}') * \hat{U}(\hat{w}) = \hat{U}(\hat{w} + \hat{w}'),$$

$$\hat{U}(\hat{w}) * \hat{U}(-\hat{w}) = \hat{1}, \ \hat{w} \in \hat{F}. \tag{3.21}$$

Their most direct realization of the isotransformation groups is that via isoexponentiation (3.3),

$$\hat{U}(\hat{w}) = \prod_k e_{\xi}^{\, i \hat{w}_k * X_k} = \prod_k e_{\xi}^{\, i X_k * \hat{w}_k} =$$

$$= \hat{1}\{\prod_k e^{iw_k T X_k}\} = \{\prod_k e^{iX_k T w_k}\}\hat{1}, \quad (3.22)$$

where the X's and w's are the infinitesimal generator and parameters, respectively, of the original algebra L. Equations (3.22) hold for some open neighborhood of N of the isoorigin of $\hat{L}$ and, in this way, characterize some open neighborhood of the isounit of $\hat{G}$. Then the isotransformations can be reduced to an ordinary transform for computational convenience,

$$x' = \hat{U} * x = \{\prod_k e_{\xi}^{i X_k * w_k}\} * x = \{\prod_k e^{iX_k T w_k}\} x, \quad (3.23)$$

with the understanding that, on rigorous mathematical grounds, only the isotransform is correct.

Still another important result obtained in [47] is the proof that conventional group composition laws admit a consistent isotopic lifting, resulting in the following *isotopy of the Baker-Campbell-Hausdorff Theorem*

$$\{e_{\xi}^{X}\} * \{e_{\xi}^{X}\} = e_{\xi}^{X_3}, X_3 = X_1 + X_2 + [X_1 \hat{,} X_2]/2 +$$

$$+ [(X_1 - X_2) \hat{,} [X_1 \hat{,} X_2]] / 12 + \ldots \quad (3.24)$$

Note the crucial appearance of the isotopic element $T(x, \dot{x}, \ddot{x}, ...)$ in the exponent of the isogroup. This ensures a structural generalization of Lie's theory of the desired nonlinear, nonlocal and noncanonical form. For details see [49] and [74].

The structure theory of isogroups is also vastly unexplored at this writing. In the following we shall point out that the conventional structure theory of Lie groups does indeed admit a consistent isotopic lifting. The isotopies of the notions of weak and strong continuity of [22] are a necessary pre-requisite. Let $\hat{L}$ be a (finite-dimensional) Lie-Santilli algebra with (ordered) basis $\{X_k\}$, $k = 1, 2, ..., N$. For a sufficiently small neighborhood N of the isoorigin of $\hat{L}$, a generic element of $\hat{G}$ can be written

$$\hat{U}(w) = \prod^{*}_{k=1,2,...,N} e_{\xi}^{iX_k w_k}, \quad (3.25)$$

which characterizes some open neighborhood M of the isounit $\hat{1}$ of $\hat{G}$. The map

$$\Phi_{\hat{U}_1}(\hat{U}_2) = \hat{U}_1 * \hat{U}_2 * \hat{U}_1^{-1}, \quad (3.26)$$

for a fixed $\hat{U}_1 \in \hat{G}$, characterizes an *inner isoautomorphism* of $\hat{G}$ onto $\hat{G}$. The corresponding isoautomorphism of the algebra $\hat{L}$ can be readily computed by considering the above expression in the neighborhood of the isounit $\hat{1}$. In fact, we have

$$\hat{U}'_2 = \hat{U}_1 * \hat{U}_2 * \hat{U}_1^{-1} \approx \hat{U}_2 + w_1 w_2 [X_2 \hat{,} X_1] + O^{(2)}. \quad (3.27)$$

The reduction of the isogroups to isoalgebras requires the knowledge of isodifferentials $\hat{\partial} w = T dw$ and isoderivatives $\hat{\partial}/\hat{\partial} w = \hat{1} dw$, under which we have the following expression in one dimension:

$$i^{-1} \frac{\hat{\partial}}{\hat{\partial} w} \hat{U}_{|w=0} = X * e_{\xi}^{iwX}{}_{|w=0} = X. \quad (3.28)$$

where we have used the isodifferential $\hat{\partial} w_k = T_k^i dw_i$ and related isoderivative (Sect. 2.C).

Thus, to every inner isoautomorphism of $\hat{G}$, there corresponds an inner isoautomorphism of $\hat{L}$ which can be expressed in the form:

$$(\hat{L})_i^j = \hat{C}_{ki}^j w^k. \quad (3.29)$$

The isogroup $\hat{G}_a$ of all inner isoautomorphism of $\hat{G}$ is called the *isoadjoint group*. It is possible to prove that the Lie-Santilli algebra of $\hat{G}_a$ is the isoadjoint algebra $\hat{L}_a$ of $\hat{L}$. This establishes that the connections between algebras and groups carry over in their entirety under isotopies.

We mentioned before that the direct sum of isoalgebras is the conventional operation because the addition is not lifted under isotopies (otherwise there will be the loss of distributivity, see [59]). The corresponding operation for groups is the semidirect product which, as such, demands care in its formulation.

Let $\hat{G}$ be an isogroup and $\hat{G}_a$ the group of all its inner isoautomorphisms. Let $\hat{G}^o_a$ be a subgroup of $\hat{G}_a$, and let $\hat{A}(\hat{g})$ be the image of $\hat{g} \in \hat{G}$ under $\hat{G}^o_a$. The *semidirect isoproduct* $\hat{G} \hat{\times} \hat{G}^o_a$ of $\hat{G}$ and $\hat{G}^o_a$ is the isogroup of all ordered pairs

$$(\hat{g}, \hat{\Lambda}) * (g', \hat{\Lambda}) = (\hat{g} * \hat{\Lambda}(\hat{g}'), \hat{\Lambda} * \hat{\Lambda}'), \quad (3.30)$$

with total isounit given by $(\hat{1}, \hat{1}_{\hat{\Lambda}})$ and inverse $\hat{g}, \hat{\Lambda}s)^{-1} = (\hat{\Lambda}^{-1}(\hat{g}^{-1}), \hat{\Lambda}^{-1})$. The above notion plays an important role in the isotopies of the inhomogeneous space-time symmetries outlined later on.

Let $\hat{G}_1$ and $\hat{G}_2$ be two isogroups with respective isounits $\hat{1}_1$ and $\hat{1}_2$. The *direct isoproduct* $\hat{G}_1 \hat{\otimes} \hat{G}_2$ of $\hat{G}_1$ and $\hat{G}_2$ is the isogroup of all ordered pairs $(\hat{g}_1, \hat{g}_2)$, $\hat{g}_1 \in \hat{G}_1$, $\hat{g}_2 \in \hat{G}_2$, with isomultiplication

$$(\hat{g}_1, \hat{g}_2) * (\hat{g}'_1, \hat{g}'_2) = (\hat{g}_1 * \hat{g}'_1, \hat{g}_2 * \hat{g}'_2), \quad (3.31)$$

total isounit $(\hat{1}_1, \hat{1}_2)$ and inverse $(\hat{g}_1^{-1}, \hat{g}_2^{-1})$. The isotopies of the remaining aspects of the structure theory of Lie groups can then be investigated by the interested reader.

Let $\hat{G}$ be an N-dimensional isotransformation group of Class I with infinitesimal generators X_k, $k = 1, 2, ..., N$. The *isodual Lie-Santilli group* $\hat{G}^d$ of $\hat{G}$ ([52], [53]) is the N-dimensional isogroup with generators $X_k{}^d = -X_k$ constructed with respect to the isodual isounit $\hat{1}^d = -\hat{1}$ over the isodual isofield $\hat{F}^d$. By recalling that $w \in F \Rightarrow w^d \in F^d$, $w^d = -w$, a generic element of $\hat{G}^d$ in a suitable neighborhood of $\hat{1}^d$ is therefore given by

$$\hat{U}^d(\hat{w}^d) = e_{\hat{\xi}^d}{}^{i^d \hat{w}^d *X^d} = -e_{\hat{\xi}}{}^{i\hat{w}*X} = -\hat{U}(\hat{w}). \quad (3.32)$$

The above antiautomorphic conjugation can also be defined for conventional Lie group, yielding the *isodual Lie group* G^d of G with generic elements $U^d(w^d) = e_{\xi^d}{}^{iw^d X} = -e_{\xi}{}^{iwX}$.

The symmetries significant for this paper are the following ones: the conventional form G, its isodual G^d, the isotopic form $\hat{G}$ and the isodual isotopic form $\hat{G}^d$. These different forms are useful for the respective characterization of particles and antiparticles in vacuum (exterior problem) or within physical media (interior problem).

It is hoped that the reader can see from the above elements that the entire conventional Lie's theory does indeed admit a consistent and nontrivial lifting into the covering Lie-Santilli formulation. Particularly important are the isotopies of the conventional representation theory, known as the *isorepresentation theory*, which naturally yields the most general known, nonlinear, nonlocal and noncanonical representations of Lie groups. Studies along these latter lines were initiated by Santilli with the isorepresentations of $\hat{SU}(2)$ and of $\hat{SO}(3)$ [61], by Klimyk and Santilli Klimyk [27], and others.

A classical realization of the Lie-Santilli isogroups can be formulated on the isotangent bundle $T*\hat{E}(r,\hat{\delta},\hat{R})$, $\hat{\delta} = T\delta$, with local chart $a = (r^k, p_k)$, $\mu = 1, 2, 3, 4, 5, 6$, $k = 1, 2, 3$, and isounit [11-71]

$$\hat{1}_2 = \text{diag.}(\hat{1}, \hat{T}) \quad (3.33)$$

the Hamilton-Santilli equations

$$\hat{\partial} a^\mu / \hat{\partial} t = \hat{\dot{a}}^\mu = \omega^{\mu\alpha} T_{2\alpha}{}^\nu \frac{\partial H}{\partial a^\nu}, \quad (3.34)$$

where $\omega^{\mu\alpha}$ is the familiar canonical Lie tensor. Eq.s (3.34) can be isoexponentiated and, after factorization of the isounit, can be written

$$a(t) = (\hat{e}^{t\hat{\dot{a}}\partial/\partial a^\mu}) * a(0) = (e^{t\omega^{\mu\alpha} T_{2\alpha}{}^\nu (\partial H/\partial a^\mu)\, \partial/\partial a^\alpha})\, a(0), \quad (3.35)$$

where we have ignored the factorization of the isounit in the isoexponent for simplicity.

An operator realization of the Lie-Santilli isogroups is given by *isounitary transformations* $x' = \hat{U} * x$ on an isohilbert space $\mathcal{H}$ [11-71] with

$$\hat{U} * \hat{U}^\dagger = \hat{U}^\dagger * \hat{U} = \hat{1}, \quad (3.36)$$

with realization via an *isohermitean operator* H

$$\hat{U} = \hat{e}^{iHt} = (e^{iHTt})\hat{1}. \quad (3.37)$$

The above classical and operator realizations are also interconnected in a unique and unambiguous way by the isoquantzation (Sect. 2.G).

3.D. Santilli's fundamental theorem on isosymmetries. We are now equipped to review without proof the following important result [52], [61] and [62]:

Theorem 3.5. *Let G be an N-dimensional Lie group of isometries of an m-dimensional metric or pseudo-metric space S(x,g,F) over a field F*

$$G: x' = A(w)\ x,\ (x'-y')^\dagger A^\dagger g A (x-y) = (x-y)^\dagger g (x-y),$$
$$A^\dagger g A = A g A^\dagger = g. \quad (3.38)$$

Then the infinitely possible isotopies $\hat{G}$ of G of Class III characterized by the same generators and parameters of G and new isounits $\hat{1}$ (isotopic elements T), automatically leave invariant the iso-composition on the isospaces $\hat{S}(x,\hat{g},\hat{F})$, $\hat{g} = Tg$, $\hat{1} = T^{-1}$,

$$\hat{G}: x' = \hat{A}(w) * x,\ (x'-y')^\dagger * \hat{A}^\dagger \hat{g} \hat{A} * (x-y) =$$

$$= (x-y)^\dagger \hat{g} (x-y),\ \hat{A}^\dagger \hat{g} \hat{A} = \hat{A} \hat{g} \hat{A}^\dagger = \hat{1} \hat{g} \hat{1}, \quad (3.39)$$

The "direct universal" of the resulting isosymmetries for all infinitely possible isotopies $g \rightarrow \hat{g}$ is then evident owing to the completely unrestricted functional dependence of the isotopic element T in the isometric $\hat{g} = Tg$. One should also note the insufficiency of the so-called *trivial isotopy*

$$X_k \rightarrow X'_k = X_k \hat{1}, \quad (3.40)$$

for the achievement of the desired form-invariance. In fact, under the above mapping the isoexponentiation becomes

$$e_{\xi}^{\ i X'_k * w_k} = \{ e^{iX'_k T w_k} \} \hat{1} = \{ e^{iX_k w_k} \} \hat{1}, \quad (3.41)$$

namely, we have the disappearance precisely of the isotopic element T in the exponent which provides the invariance of the isoseparation.

3.E. Isotopies and isodualities of the rotational symmetry. We now illustrate the Lie-Santilli isotheory with the first mathematically and physically significant case, the *isotopies of the rotational symmetry*, also called *isorotational symmetry*. They were first achieved in [53] and then studied in details in [61] and [62], including the isotopies of SU(2), their isorepresentations, the iso-Clebsh-Gordon coefficients, etc.

Consider the lifting of the perfect sphere in Euclidean space $E(r,\delta,\Re)$ with local coordinates $r = (x, y, z)$, and metric $\delta = \text{diag.}(1, 1, 1)$ over the reals $\Re$,

$$r^2 = r^t \delta r = x x + y y + z z, \quad (3.42)$$

into the most general possible ellipsoid of Class III on isospace $\hat{E}^{III}(r,\hat{\delta},\hat{\Re})$, $\hat{\delta} = T\delta$, $T = \text{diag.}(g_{11}, g_{22}, g_{33})$, $\hat{1} = T^{-1}$.

$$r^2 = r^t \hat{\delta} r = x g_{11} y + y g_{22} y + z g_{33} z,$$

$$\delta^\dagger = \hat{\delta},\ \hat{g}_{kk} = g_{kk}(t, r, \dot{r}, \ddot{r}, ...) \neq 0, \quad (3.43)$$

The invariance of the original separation r^2 is the conventional rotational symmetry O(3). The isotopic techniques then permit the construction, in the needed explicit and finite form, of the isosymmetries $\hat{O}(3)$ of all infinitely possible generalized invariants r^2 via the following steps: (1) Identification of the basic isotopic element T in the lifting $\delta \rightarrow \hat{\delta} = T\delta$ which, in this particular case, is given by the new metric $\hat{\delta}$ itself, $T = \hat{\delta}$, and identification of the fundamental unit of the theory, $\hat{1} = T^{-1}$; (2) Consequential lifting of the basic field $\Re(n,+,\times) \Rightarrow \hat{\Re}(\hat{n},+,*)$; (3) Identification of the isospace in which the generalized metric $\hat{\delta}$ is defined, which is given by the three-dimensional isoeuclidean spaces $\hat{E}(r,\hat{\delta},\hat{\Re})$, $\hat{\delta} = T\delta$, $\hat{1} = T^{-1}$; (4) Construction of the $\hat{O}(3)$ symmetry via the use of the original parameters of O(3) (the Euler's angles θ_k, k = 1, 2, 3), the original generators (the angular momentum components $M_k = \epsilon_{kij} r^i p_j$) in their fundamental (adjoint) representation, and the new metric $\hat{\delta}$; and (5) Classification, interpretation and application of the results.

The explicit construction of $\hat{O}(3)$ is straightforward. According to the Lie-Santilli theory, the connected component $S\hat{O}(3)$ of $\hat{O}(3)$ is given by [53]

$$S\hat{O}(3):\ r' = \hat{R}(\theta) * r,\ \hat{R}(\theta) = \prod{}^*_{k=1,2,3} e_{\xi}^{\ iM_k\theta_k} =$$

$$= \{\prod_{k=1,2,3} e^{iM_k T\theta_k}\} \hat{1}, \quad (3.44)$$

while the discrete component is given by the *isoinversions* [loc. cit.] $r' = \hat{\pi} * r = \pi r = -r$, where π is the conventional inversion.

Under the assumed conditions on the isotopic element T, the convergence of isoexponentiations is ensured by the original convergence, thus permitting the explicit construction of the isorotations, with example around the third axis [53]

$$x' = x \cos[\theta_3 (g_{11} g_{22})^{\frac{1}{2}}] +$$

$$+ y\, g_{22} (g_{11} g_{22})^{-\frac{1}{2}} \sin[\theta_3 (g_{11} g_{22})^{\frac{1}{2}}],$$

$$y' = -x\, g_{11} (g_{11} g_{22})^{-\frac{1}{2}} \sin[\theta_3 (g_{11} g_{22})^{\frac{1}{2}}] +$$

$$+ y \cos[\theta_3 (g_{11} g_{22})^{\frac{1}{2}}],$$

$$z' = z. \qquad (3.45)$$

(see [61b] for general isorotations). One should note that the argument of the trigonometric functions as derived via the above isoexponentiation coincides with the isoangle of the isotrigonometry in $\hat{E}(r,\hat{\delta},\hat{\Re})$ (see paper [60]) thus confirming the remarkable compatibility and interconnections of the various branches of the the isotopic theory.

The computation of the isoalgebras $\hat{o}(3)$ of $\hat{O}(3)$ is then straighforward [53]. In fact, when M_k are assumed to be in their regular representation we have [53]

$$\hat{o}(3) : [M_i \hat{,} M_j] = M_i T M_j - M_j T M_i = \hat{C}_{ij}{}^k * M_k, \quad (3.46)$$

where $\hat{C}_{ij}{}^k = \epsilon_{ijk}\, g_{kk}{}^{-1}\, \hat{1}$. The above isoalgebra illustrates the explicit dependence of the structure *functions*. The proof of the isomorphism $\hat{o}(3) \approx o(3)$ was done [loc. cit.] via a suitable reformulation of the basis under which the structure functions recover the value $\hat{\epsilon}_{ijk} = \epsilon_{ijk} \hat{1}$.

The isocenter of $s\hat{o}(3)$ is characterized by the *isocasimir invariants*

$$C^{(0)} = \hat{1}, \quad C^{(2)} = M^2 = M * M = \sum_{k=1,2,3} M_k T M_k. \quad (3.47)$$

In hadronic mechanics [61] one of the possible realizations is the following. The linear momentum operator has the isotopic form

$$p_k * |\hat{\psi}> = -i \hat{\nabla}_k |\hat{\psi}> = -i \hat{1}_k{}^i \nabla_i |\hat{\psi}>.$$

(see [II-71] for a different realization). The fundamental isocommutation rules are then given by

$$[r^i \hat{,} p_j] = i\,\delta^i{}_j = i \hat{1}\, \delta^i{}_j, \quad [r^i \hat{,} r^j] = [p_i \hat{,} p_j] = 0.$$

However, in their contravariant form the coordinates are given by $r_k = \hat{\delta}_{ki}\, r^i$. As a result $\hat{\nabla}_i r_j = \delta_{ij}$ (where the delta is the *conventional* Kronecker delta). In this case the fundamental isocommutation rules are given by

$$[r_i \hat{,} p_j] = i\,\delta^i{}_j = i \hat{1}\, \delta^i{}_j, \quad [r_i \hat{,} r_j] = [p_i \hat{,} p_j] = 0,$$

namely, their eigenvalues *coincide* with the quantum ones. The operator isoalgebra $\hat{o}(3)$ with generators $M_k = \epsilon_{kij}\, r_i\, p_j$ is then given by

$$\hat{o}(3) : \quad [M_i \hat{,} M_j] = M_i T M_j - M_j T M_i = i\, \hat{\epsilon}_{ij}{}^k * M_k,$$

where $\hat{\epsilon}_{ij}{}^k = \epsilon_{ijk} \hat{1}$, namely *the product of the algebra is generalized, but the structure constants are the conventional ones* (see [61] for details). The above results illustrates again the abstract identity of quantum and hadronic mechanics.

Note the nonlinear-nonlocal-noncanonical character of isotransformations (3.45) owing to the unrestricted functional dependence of the diagonal elements g_{kk}. Note also the extreme simplicity of the final results. In fact, the explicit symmetry transformations of separation (3.43) are provided by just plotting the given g_{kk} values into transformations (3.45) without any need of any additional computation. Note finally that the above invariance includes as particular case the general isosymmetry $\hat{O}(3)$ of (the space-component of) gravitation which, since it is locally Euclidean, remains isomorphic to O(3).

As an example, the symmetry of the space-component of the Schwartzschild line element is given by plotting the following values

$$g_{11} = (1 - M/r)^{-1}, \quad g_{22} = r^2, \quad g_{33} = r^2 \sin^2\theta, \quad (3.48)$$

(see next section for the full (3+1)-dimensional case).

Despite this simplicity, the implications of the above results are nontrivial. On physical grounds, the isounit $\hat{1} > 0$ permits a direct representation of the nonspherical shapes, as well as all their infinitely possible deformations. By recalling that O(3) is a *theory of rigid bodies*, $\hat{O}(3)$ results to be a *theory of deformable bodies* [53] with fundamentally novel physical applications in the theory of elasticity, nuclear physics, particle physics, crystallography, and other fields ([61], [62]).

On mathematical grounds, we have equally intriguing novel insights. To see them, one must first understand the background isogeometry $\hat{E}^{III}(r,\delta,\Re)$ which unifies all possible conics in $E(r,\delta,\Re)$ [61a], as mentioned earlier. To be explicit in this important point, the geometric differences between (oblate or prolate) ellipsoids and (elliptic or hyperbolic) paraboloids have mathematical sense when projected in our Euclidean space $E(r,\delta,\Re)$. However all these surfaces are geometrically unified with the perfect isosphere in $\hat{E}(r,\hat{\delta},\hat{\Re})$.

These geometric occurrences permits the unification of O(3) and O(2.1), as well as of all their infinitely possible isotopes. In fact, the classification of all possible isosymmetries $\hat{O}(3)$, achieved in the original derivation [53], includes:

(1) The compact O(3) symmetry evidently for $\hat{\delta} = \delta$ = diag. (1, 1, 1);

(2) The noncompact O(2.1) symmetry evidently for $\hat{\delta}$ = diag. (1, 1, −1);

(3) The isodual $O^d(3)$ of O(3) holding for $\hat{\delta}$ = diag. (−1, −1, −1);

(4) The isodual $O^d(2.1)$ of O(2.1) holding for $\hat{\delta}$ = diag. (−1, −1, 1);

(5) The infinite family of compact isotopes $\hat{O}(3) \approx O(3)$ with $\hat{1} > 0$ for $\hat{\delta}$ = diag. (b_1^2, b_2^2, b_3^2), $b_k > 0$;

(6) The infinite family of noncompact isotopes $\hat{O}(2.1) \approx O(2.1)$ for $\hat{\delta}$ = diag. $(b_1^2, b_2^2, -b_3^2)$;

(7) The infinite family of compact isodual isotopes $\hat{O}^d(3) \approx O^d(3)$ for $\hat{\delta}$ = diag. $(-b_1^2, -b_2^2, -b_3^2)$;

(8) The infinite family of isodual isotopes $\hat{O}^d(2.1) \approx O^d(2.1)$ for $\hat{\delta}$ = diag. $(-b_1^2, -b_2^2, b_3^2)$.

Even greater differentiations between the Lie and Lie-Santilli theories occur in their representations because of the change in the eigenvalue equations due to the nonunitarity of the map indicated in Sect. 1, from the familiar form $H\psi = E^\circ\psi$, to the isotopic form $\hat{H}\hat{*}\hat{\psi} = \hat{E}\hat{*}\hat{\psi} \equiv E\hat{\psi}$, $E^\circ \neq E$), thus implying generalized weights, Cartan tensors and other structures studied earlier.

The first differences emerge in the spectrum of eigenvalues of $\hat{o}(2)$ and $o(2)$. In fact, the $o(2)$ algebra on a conventional Hilbert space *solely* admits the spectrum M = 0, 1, 2, 3 (as a necessary condition of unitarity). For the covering $\hat{o}(2)$ isoalgebra on an isohilbert space with isotopic element $T = \text{Diag.}(g_{11}, g_{22})$, the spectrum is instead given by $\hat{M} = g_{11}^{-1/2} g_{22}^{-1/2} M$ and, as such, it can acquire *continuous* values in a way fully consistent with the condition, this time, of isounitarity. For the general $\hat{O}(3)$ case see also the detailed studied of ref.s [61].

Similarly, the unitary irreducible representations of su(2) are characterize the familiar eigenvalues

$$J_3 \hat{\psi} = M\psi, \quad J^2\psi = J(J+1)\psi, \quad M = J, J-1, \ldots, -J, \quad J = 0, \tfrac{1}{2}, 1, \ldots \quad (3.49)$$

Three classes of irreducible isorepresentation of $\hat{su}(2)$ were identified in [63] which, for the adjoint case, are given by the following generalizations of Pauli's matrices:

(1) *Regular isopauli matrices*

$$\hat{\sigma}_1 = \Delta^{-\frac{1}{2}}\begin{pmatrix} 0 & g_{11} \\ g_{22} & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \Delta^{-\frac{1}{2}}\begin{pmatrix} 0 & -i\,g_{11} \\ +i\,g_{22} & 0 \end{pmatrix},$$

$$\hat{\sigma}_3 = \Delta^{-\frac{1}{2}}\begin{pmatrix} g_{22} & 0 \\ 0 & -g_{11} \end{pmatrix}. \quad (3.50a)$$

$T = \text{diag.}(g_{11}, g_{22})$, $\Delta = \det T = g_{11} g_{22} > 0$,

$$[\hat{\sigma}_i, \hat{\sigma}_j]_{\hat{}} = i\, 2\Delta^{\frac{1}{2}} \epsilon_{ijk} \hat{\sigma}_k. \quad (3.50b)$$

$$\hat{\sigma}_3 \hat{*} |\hat{b}> = \pm \Delta^{\frac{1}{2}} |\hat{b}>, \quad \hat{\sigma}^2 \hat{*} |\hat{b}> = 3\Delta |\hat{b}>. \quad (3.50c)$$

(2) *Irregular isopauli matrices*

$$\hat{\sigma}_1' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1, \quad \hat{\sigma}_2' = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} = \sigma_2,$$

$$\hat{\sigma}_3' = \begin{pmatrix} g_{22} & 0 \\ 0 & -g_{11} \end{pmatrix} = \Delta\hat{1}\sigma_3, \quad (3.51a)$$

$$[\hat{\sigma}'_1 \hat{,} \hat{\sigma}_2']_{\hat{}} = 2\,i\,\hat{\sigma}_3',$$
$$[\hat{\sigma}'_2, \hat{\sigma}_3']_{\hat{}} = 2\,i\,\Delta\,\hat{\sigma}_1', \qquad [\hat{\sigma}_3', \hat{\sigma}_1']_{\hat{}} = 2\,i\,\Delta\,\hat{\sigma}_2', \tag{3.51b}$$
$$\hat{\sigma}'_3 * |\hat{b}> = \pm \Delta |\hat{b}>,$$
$$\hat{\sigma}^{\hat{2}*} |\hat{b}> = \Delta(\Delta+2)|\hat{b}>. \tag{3.51c}$$

(3) *Standard isopauli matrices*

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & \lambda \\ \lambda^{-1} & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i\lambda \\ i\lambda^{-1} & 0 \end{pmatrix},$$
$$\hat{\sigma}_3 = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & -\lambda \end{pmatrix}, \tag{3.52a}$$

$$T = \mathrm{diag.}(\lambda, \lambda^{-1}), \quad \lambda \neq 0, \qquad \Delta = \det T = 1,$$
$$[\hat{\sigma}''_i, \hat{\sigma}''_j]_{\hat{}} = i\,\epsilon_{ijk}\,\hat{\sigma}''_k, \tag{3.52b}$$
$$\hat{\sigma}''_3 * |\hat{b}> = \pm |\hat{b}>, \qquad \hat{\sigma}''^{\hat{2}*} |\hat{b}> = 3|\hat{b}>. \tag{3.52c}$$

The primary differences in the above isorepresentations are the following. For the case of the regular isorepresentations, the isotopic contributions can be factorized with respect to the conventional Lie spectrum. For the irregular case this is no longer possible. Finally, for the standard case we have conventional spectra of eigenvalues under a generalized structure of the matrix representations, as indicated by the appearance of a completely unrestricted, integro-differential function λ.

The regular and irregular representations of $\hat{o}(3)$ and $\hat{su}(2)$ are applied to the angular momentum and spin of particles under extreme physical conditions, such as an electron in the core of a collapsing star. The standard isorepresentations are applied to conventional particles evidently because of the preservation of conventional quantum numbers. The appearance of the isotopic degrees of freedom then permit novel physical applications, that is, applications beyond the capacity of Lie's theory even for the simpler case of preservation of conventional spectra (see Section 3.G).

The spectrum-preserving map from the conventional representations J_g of a Lie-algebra L with metric tensor g to the covering isorepresentations $\mathfrak{J}_{\hat{g}}$ of the Lie-Santilli algebra $\hat{L}$ with isometric $\hat{g} = Tg$ and isounit $\hat{1} = T^{-1}$ is important for physical application. It is called the *Klimyk rule* [27] and it given by

$$\mathfrak{J}_{\hat{g}} = J_g\,P, \qquad P = k\,\hat{1}, \quad k \in \hat{F}, \tag{3.53}$$

under which Lie algebras are turned into Lie-Santilli isoalgebras

$$J_i J_j - J_j J_i = C_{ij}^{\ k} J_k \;\Rightarrow\; (\mathfrak{J}_i * \mathfrak{J}_j - \mathfrak{J}_j * \mathfrak{J}_i)\,k^{-1}\,T =$$
$$= C_{ij}^{\ k}\,k^{-1}\,T\,\mathfrak{J}_k,$$

that is,

$$\mathfrak{J}_i * \mathfrak{J}_j - \mathfrak{J}_j * \mathfrak{J}_i = C_{ij}^{\ k}\,\mathfrak{J}_k,$$

thus showing the preservation of the original structure constants.

However, by no means, the Klimyk rule can produce *all* Lie-Santilli isoalgebras, because the latter are generally characterized by *nonunitary* transforms of conventional algebras, with a general variation of the structure constants.

Nevertheless, the Klimyk rule is sufficient for a number of physical applications where the preservation of conventional quantum numbers is important, because it permits the identification of one specific and explicit form of standard isorepresentations with "hidden" degrees of freedom represented by the isotopic element T available for specific uses. For instance, the standard isopauli matrices permit the reconstruction of the exact isospin symmetry in nuclear physics under electromagnetic and weak interactions [63], or the construction of the isoquark theory with all conventional quantum numbers, yet an *exact confinement* (with an identically null probability of tunnel effects for free quarks because of the incoherence between the interior and exterior Hilbert spaces) [68], and other novel applications.

3.F. Isotopies and isodualities of the Lorentz and Poincare´ symmetries.

Consider the line element in Minkowski space $x^2 = x^\mu \eta_{\mu\nu} x^\nu$, $\mu, \nu = 1, 2, 3, 4$, with local coordinates $x = \{x^1, x^2, x^3, x^4\}$, $x^4 = c_0 t$, and metric $\eta = \mathrm{diag.}(1, 1, 1, -1)$. Its simple invariance group, the six-dimensional Lorentz group L(3.1), is characterized by the (ordered sets of) parameters given by the Euler's angles and speed parameter, $w = \{ w_k \} = \{ \theta, v \}$, $k = 1, 2, ..., 6$, and generators $X = \{ X_k \} = \{ M_{\mu\nu} \}$, in their known fundamental representation (see, e.g., [31], [32]).

Suppose now that the Minkowskian line element is lifted into the most general possible nonlinear-integral form verifying the conditions of Class III

$$x^2 = x^\mu \hat{g}_{\mu\nu}(x, \dot{x}, \ddot{x}, ...) x^\nu, \quad \det \hat{g} \neq 0, \quad \hat{g} = \hat{g}^\dagger, \tag{3.54}$$

which represent: all modifications of the Minkowski metric as encountered, e.g., in particle physics; conventional exterior gravitational line elements with $\hat{g} = \hat{g}(x)$, such as the full Schwartzschild line element; all its possible generalizations for the interior problem; etc.

The explicit form of the simple, six-dimensional invariance of generalized line element x^2 was first constructed by Santilli [51] by following the space-time version of Steps 1 to 5 of the preceding section. Step 1 is the identification of the fundamental isotopic element T via the factorization of the Minkowski metric, $\hat{g} = T\eta$ which, under the assumed conditions, can always be diagonalized into the form

$$T = \text{diag.}(g_{11}, g_{22}, g_{33}, g_{44}), \quad T = T^\dagger, \quad \det T \neq 0 \tag{3.55}$$

The fundamental isounit of the theory is then given by $\hat{1} = T^{-1}$.

Step 2 is the lifting of the conventional numbers into the isonumbers via the isofields $\hat{\mathfrak{R}}(\hat{n},+,*)$, $\hat{n} = n\hat{1}$ (which are different than those of $\hat{O}(3)$ because of the different dimension of the isounit).

Step 3 is the construction of the isospaces in which the isometric $\hat{g}$ is properly defined, which are given by the isominkowski spaces $\hat{M}(x,\hat{g},\hat{\mathfrak{R}})$. The reader should keep in mind that, when $\hat{g}$ is a conventional Riemannian metric, isospaces $\hat{M}(x,\hat{g},\hat{\mathfrak{R}})$ are not the Riemannian spaces $R(x,\hat{g},\mathfrak{R})$ because the basic units of the two spaces are different.

Step 4 is also straightforward. The *Lorentz-Santilli isosymmetry* $\hat{L}(3.1)$ is characterized by the isotransformations

$$\hat{O}(3.1): \quad x' = \hat{\Lambda}(\hat{w}) * x = \tilde{\Lambda}(w)\, x, \tag{3.56}$$

verifying the basic properties

$$\hat{\Lambda}^\dagger \hat{g} \hat{\Lambda} = \hat{\Lambda} \hat{g} \hat{\Lambda}^\dagger = \hat{1} \hat{g} \hat{1}, \quad \text{or}$$

$$\tilde{\Lambda}^\dagger \hat{g} \tilde{\Lambda} = \tilde{\Lambda} \hat{g} \tilde{\Lambda}^\dagger = \hat{g}, \tag{3.57a}$$

$$\hat{\text{Det}}\, \hat{\Lambda} = [\, \text{Det}\, (\hat{\Lambda}\, T)\,] = \pm \hat{1}. \tag{3.57b}$$

It is easy to see that $\hat{L}(3.1)$ preserves the original connectivity properties of L(3.1) (see [61] for a detailed study). The connected component $S\hat{O}(3.1)$ of $\hat{L}(3.1)$ is characterized by $\hat{\text{Det}}\, \hat{\Lambda} = +\hat{1}$ and has the structure [loc. cit.]

$$\hat{\Lambda}(w) = \prod{}^*_{k=1,2,...,6}\, e_\xi^{\, i X_k * \hat{w}_k} =$$

$$= \{ \prod_{k=1,2,...,6} e^{\, i X_k T w_k} \} \hat{1}, \tag{3.58}$$

where the parameters are the conventional ones, the generators X_k are also the conventional ones in their fundamental representation and the isotopic element T is given by Equations (3.23). The discrete part of $\hat{L}(3.1)$ is characterized by $\hat{\text{Det}}\, \hat{\Lambda} = -\hat{1}$, and it is given by the *space-time isoinversions* [loc. cit.]

$$\hat{\pi} * x = \pi x = -r, x^4), \quad \hat{\tau} * x = \tau x = (r, -x^4). \tag{3.59}$$

Again, under the assumed conditions for T, the convergence of infinite series (3.58) is ensured by the original convergence, thus permitting the explicit calculation of the symmetry transformations in the needed explicit, finite form. Their space components have been given in the preceding Section 3.E. The additional *Lorentz-Santilli isoboosts* can also be explicitly computed, yielding the expression for all possible isometrics $\hat{g}$ [51]

$$x'^1 = x^1, \quad x'^2 = x^2, \tag{3.60a}$$

$$x'^3 = x^3 \cosh [\, v\, (g_{33}\, g_{44})^{\frac{1}{2}}\,] -$$

$$- x^4 g_{44} (g_{33} g_{44})^{-\frac{1}{2}} \sinh [v (g_{33} g_{44})^{\frac{1}{2}}] =$$

$$= \hat{\gamma} (x^3 - g_{33}^{-1/2} g_{44}^{1/2} \beta x^4), \quad (3.60b)$$

$$x'^4 = - x^3 g_{33} (g_{33} g_{44})^{-\frac{1}{2}} \sinh [v (g_{33} g_{44})^{\frac{1}{2}}] +$$

$$+ x^4 \cosh [v (g_{33} g_{44})^{\frac{1}{2}}] =$$

$$= \hat{\gamma} (x^4 - g_{33}^{1/2} g_{44}^{-1/2} \beta x^3), \quad (3.60c)$$

where $x^4 = c_o t$, $\beta = v / c_o$,

$$\beta = v^k g_{kk} v^k / c_o g_{44} c_o, \quad (3.61a)$$

$$\cosh [v (g_{33} g_{44})^{\frac{1}{2}}] = \hat{\gamma} = (1 - \beta^2)^{-\frac{1}{2}},$$

$$\sinh [v (g_{33} g_{44})^{\frac{1}{2}}] = \beta \hat{\gamma}. \quad (3.61b)$$

Again, one should note: (A) the unrestricted character of the functional dependence of the isometric $\hat{g}$; (B) the remarkable simplicity of the final results where by the explicit symmetry transformations are merely given by plotting the values $g_{\mu\mu}$ in Equations (3.60); and (C) the generally nonlinear-nonlocal-noncanonical character of the isosymmetry.

The isocommutation rules when the generators $M_{\mu\nu}$ are in their regular representation can also be readily computed and are given by [loc. cit.]

$$\hat{o}(3.1): [M_{\mu\nu} \hat{,} M_{\alpha\beta}] = \hat{g}_{\nu\alpha} M_{\beta\mu} - \hat{g}_{\mu\alpha} M_{\beta\nu} -$$

$$- \hat{g}_{\nu\beta} M_{\alpha\mu} + \hat{g}_{\mu\beta} M_{\alpha\nu}, \quad (3.62)$$

with isocasimirs

$$C^{(0)} = 1, \quad C^{(1)} = \tfrac{1}{2} M_{\mu\nu} T M^{\mu\nu} = M * M - N * N, \quad (3.63a)$$

$$C^{(3)} = \tfrac{1}{4} \epsilon^{\mu\nu\rho\sigma} M_{\mu\nu} T M_{\rho\sigma} = - M * N, \; M =$$

$$= \{ M_{12}, M_{23}, M_{31} \}, \quad N = \{ M_{01}, M_{02}, M_{03} \}. \quad (3.63c)$$

The classification of all possible isotopes $S\hat{O}(3.1)$ was also done in the original construction [51] via the realizations of the isotopic element

$$T = \text{diag.} (\pm b_1^2, \pm b_2^2, \pm b_3^2, \pm b_4^2), \quad b_\mu > 0, \quad (3.64)$$

where the b's are the characteristic functions of the interior medium, resulting in:

(1) The conventional orthogonal symmetry SO(4) for T = diag. (1, 1, 1, −1);

(2) The conventional Lorentz symmetry SO(3.1) for T = diag. (1, 1, 1, 1);

(3) the conventional de Sitter symmetry SO(2.2) for T = diag. (1, 1, −1, 1);

(4) the isodual $SO^d(4)$ for T = diag. (−1, −1, −1, 1);

(5) the isodual $O^d(3.1)$ for T = − diag. (1, 1, 1, 1);

(6) the isodual $SO^d(2.2)$ for T = diag. (−1, −1, 1, −1);

(7) the infinite family of isotopes $S\hat{O}(4) \approx SO(4)$ for $T = \text{diag.} (b_1^2, b_2^2, b_3^2, -b_4^2)$;

(8) the infinite family of isotopes $S\hat{O}(3.1) \approx SO(3.1)$ for $T = \text{diag.} (b_1^2, b_2^2, b_3^2, b_4^2)$;

(9) the infinite family of isotopes $S\hat{O}(2.2) \approx SO(2.2)$ for $T = \text{diag.} (-b_1^2, b_2^2, b_3^2, b_4^2)$;

(10) the infinite family of isoduals $S\hat{O}^d(4) \approx SO^d(4)$ for $T = \text{diag.} (-b_1^2, -b_2^2, -b_3^2, b_4^2)$;

(11) the infinite family of isoduals $S\hat{O}^d(3.1) \approx SO(3.1)$ for $T = - \text{diag.} (b_1^2, b_2^2, b_3^2, b_4^2)$;

(12) the infinite family of isoduals $S\hat{O}^d(2.2) \approx SO^d(2.2)$ for $T = \text{diag.} (b_1^2, -b_2^2, -b_3^2, -b_4^2)$.

On the basis of the above results, Santilli [61] submitted the *conjecture* that *all simple Lie algebra of the same dimension over a field of characteristic zero in Cartan classification can be unified into one single abstract isotopic algebra of the same dimension.*

The above conjecture was proved by Santilli for the cases n = 3 and 6. A theorem unifying all possible fields into the isoreals was proved by Kadeisvili et al [26] in the expectation of such general unification, but its study has remained unexplored at this writing.

In the above presentation we have shown that the lifting of the Lorentz symmetry can be naturally formulated for Class III. Nevertheless, whenever dealing with physical applications, the isotopic element is restricted to have the positive- or negative-definite structure $T = \pm\text{diag.} (b_1^2, b_2^2, b_3^2, b_4^2)$, thus restricting the isotopies to $S\hat{O}(3.1) \approx SO(3.1)$ and $S\hat{O}^d(3.1) \approx SO^d(3.1)$.

The operator realization of the latter Lorentz-Santilli isoalgebra is the following. The linear fourmomentum admits the isotopic realization [II-71]

$$p_\mu * |\hat\psi> = -i\partial_\mu|\hat\psi> = -iT_\mu{}^\nu\partial_\nu|\hat\psi>.$$

Also, for $x_\mu = \eta_{\mu\nu}x^\nu$ (where η is the conventional Minkowski metric). one can show that $\partial_\mu x_\nu = \hat\eta_{\mu\nu}$. The fundamental relativistic isocommutation rules are then given by ([61], [65])

$$[x_\mu \hat{,} p_\nu] = i\hat\eta_{\mu\nu}, \quad [x_\mu \hat{,} x_\nu] = [p_\mu \hat{,} p_\nu] = 0,$$

The isocommutation rules are then given by

$$\hat{o}(3.1): [M_{\mu\nu} \hat{,} M_{\alpha\beta}] = i(\hat\eta_{\nu\alpha}M_{\beta\mu} - \hat\eta_{\mu\alpha}M_{\beta\nu} -$$

$$-\hat\eta_{\nu\beta}M_{\alpha\mu} + \hat\eta_{\mu\beta}M_{\alpha\nu}), \qquad (3.62)$$

thus confirming the isomorphism $S\hat{O}(3.1) \approx SO(3.1)$ for all positive-definite T.

The *Poincaré-Santilli isosymmetry*

$$\hat{P}(3.1) = \hat{L}(3.1) \times \hat{T}(3.1), \qquad (3.65)$$

and its isodual $\hat{P}^d(3.1)$ have been been constructed in their classical [62] and operator [62] forms as well as in their isospinorial form $\hat{\mathcal{P}}(3.1) = S\hat{L}(2.\hat{C})\times\hat{T}(3.1)$ [69]. We here limit ourselves to a brief outline of the nonspinorial case mainly to illustrate the advances in the structure of isoalgebras and isogroups studied in this paper.

A generic element of $\hat{P}(3.1)$ can be written $\hat{A} = (\hat\Lambda, \hat{a})$, $\hat\Lambda \in \hat{O}(3.1)$, $\hat{a} \in \hat{T}(3.1)$ with isocomposition

$$\hat{A}' * \hat{A} = (\hat\Lambda', \hat{a}') * (\hat\Lambda, \hat{a}) = (\hat\Lambda * \hat\Lambda', \hat{a} + \hat\Lambda' * a'), \qquad (3.66)$$

The realization important for physical applications is that via conventional generators in their adjoint representation for a system of n particles of non-null mass m_a

$$X = \{X_k\} = \{M_{\mu\nu} = \textstyle\sum_a (x_{a\mu}p_{a\nu} - x_{a\nu}p_{a\mu}),$$

$$P = \textstyle\sum_a p_a\}, k = 1, 2, .10, \qquad (3.67)$$

and conventional parameters $w = \{w_k\} = \{v, \theta, a\}$, where v represents the Lorentz parameters, θ represents the Euler's angles, and a characterizes conventional space-time translations.

The connected component of the isopoincaré group is given by

$$\hat{P}(3.1): \quad x' = \hat{A} * x,$$

$$\hat{A} = \textstyle\prod^*{}_k e_\xi{}^{iX_k w_k} = \{\prod_k e^{iX_k T w_k}\}1, \qquad (3.68)$$

where the isotopic element T and the Lorentz generators $M_{\mu\nu}$ have the same realization as for $\hat{O}(3.1)$. The primary different with isosymmetries $\hat{O}(3.1)$ is the appearance of the isotranslations

$$\hat{T}(3.1) * x = \{e_\xi{}^{iP\eta a}\} * x = e_\xi{}^{iP\hat{g}a}\} * x = x + \hat{a},$$

$$\hat{T}(3.1) * p = 0. \qquad (3.69)$$

The *general Poincaré-Santilli isotransformations* are then given by ([61], [62])

$$x' = \hat\Lambda * x \quad \text{Lorentz-Santilli isotransf.}, \qquad (3.70a)$$

$$x' = x + a_o B(s, x, \dot{x}, \ddot{x}, \ldots), \quad \text{isotransl.}, \qquad (3.70b)$$

$$x' = \hat\pi_r * x = (-r, x^4), \quad \text{space isoinv.}, \qquad (3.70c)$$

$$x' = \hat\pi_t * x = (r, -x^4), \quad \text{time isoinv.}, \qquad (3.70d)$$

where the B-functions are given by the expansions

$$B_\mu = b_\mu + a^\alpha [b_\mu \hat{,} P_\alpha] / 1! +$$

$$+ a^\alpha a^\beta [[b_\mu \hat{,} P_\alpha] \hat{,} P_\beta] / 2! + \ldots\ldots \qquad (3.71)$$

The isocommutation rules of $\hat{P}(3.1)$ in the operator realizations indicated earlier are

$$[M_{\mu\nu} \hat{,} M_{\alpha\beta}] = i(\hat\eta_{\nu\alpha}M_{\beta\mu} - \hat\eta_{\mu\alpha}M_{\beta\nu} -$$

$$-\hat\eta_{\nu\beta}M_{\alpha\mu} + \hat\eta_{\mu\beta}M_{\alpha\nu}), \qquad (3.72a)$$

$$[M_{\mu\nu} \hat{,} P_\alpha] = i(\hat\eta_{\mu\alpha}P_\nu - \hat\eta_{\nu\alpha}P_\mu),$$

$$[P_\mu \hat{,} P_\nu] = 0, \mu, \nu, \alpha, \beta = 1, 2, 3, 4, \quad (3.72b)$$

and the isocenter is characterized by the isocasimirs

$$C^{(0)} = \hat{1}, \quad C^{(1)} = \hat{P}^2 = P\,T\,P = P_\mu \hat{g}^{\mu\nu} P_\nu, \quad (3.73a)$$

$$C^{(2)} = \hat{W}^2 = \hat{W}_\mu \hat{g}^{\mu\nu} \hat{W}_\nu, \quad (3.73b)$$

$$\hat{W}_\mu = \epsilon_{\mu\alpha\beta\rho} \mathcal{J}^{\alpha\beta} * p^\rho. \quad (3.73c)$$

The *restricted isotransformations* occur when the isotopic element T is constant.

An important application of the isotranslation is the characterization of the so-called *isoplane-waves* on $\hat{M}(x,\hat{\eta},\hat{\Re})$

$$\hat{\psi}(x) = e_{\xi}^{\;ipx} = \hat{1}\, e^{ipTx} = \hat{1}\, e^{ip_\mu \hat{g}^{\mu\nu} x_\nu} =$$

$$= \hat{1}\, e^{i(p_k b_k^2 x_k - p_4 b_4^2 x_4)}, \quad (3.74)$$

which are solutions of the isotopic field equations, represents electromagnetic waves propagating within inhomogeneous and anisotropic media such as out atmosphere and offer quite intriguing predictions for experimentally verifiable <novel> effects, that is, effects beyond the predictive or descriptive capacities of the Poincaré symmetry (see the companion paper [60]).

As one can see, the verification of total conservation laws (for a system assumed as isolated from the rest of the universe), is intrinsic in the very structure of the isosymmetry. In fact, the generators are the conventional ones and, since they are invariant under the action of the group they generate, they characterize conventional total conservation laws. The simplicity of reading off the total conservation laws from the generators of the isosymmetry should be compared with the rather complex proof in conventional gravitational theories.

The *isodual Poincaré-Santilli isosymmetry* $\hat{P}^d(3.1)$ is characterized by the isodual generators $X_k{}^d = -X_k$, the isodual parameters $w_k{}^d = -w_k$, and the isodual isotopic element $T^d = -T$, resulting in the change of sign of isotransforms. This implies a novel *law of universal invariant under isoduality* which essentially state that any system which is invariant under a given symmetry is automatically invariant under its isodual. In turn, this law apparently permits novel advances in the study of antiparticles [61].

The significanmce of the Lie-Santilli isotheory for gravitation is illustrated by the following important property of the isosymmetry $\hat{P}(3.1)$ which evidently follows from of Theorem 3.5:

Theorem 3.6 [51]. *The Poincaré-Santilli isosymmetry $\hat{P}(3.1)$ is directly universal for all infinitely possible (3+1)-dimensional invariants*

$$(x-y)^\mu\, \hat{\eta}_{\mu\nu}(x, \dot{x}, \ddot{x}, ...)\,(x-y)^\nu, \quad \hat{\eta} = T\eta, \quad (3.75)$$

Note that the above theorem includes as particular cases the conventional Riemannian metric $g(x) = \hat{\eta}(x)$, thus providing the universal invariance of *exterior* gravitation in vacuum. More generally, the theorem includes all infinitely possible signature-preserving modifications of the Minkowski and Riemannian metrics for interior problems. The simplicity of this universal invariance should also be kept in mind and compared with the known complexity of other approaches to nonlinear symmetries. In fact, one merely *plots* the $g_{\mu\mu}$ elements in isotransforms (3.45), (3.60), (3.70) without any need to compute anything, because the invariance of general separation (3.75) is ensured by the theorem. For numerous examples, see [61], [62].

As anticipated in Sect. 1.E, a remarkable property of the Lie-Santilli theory is the capability to unify in one, single, abstract isosymmetry $\hat{P}(3.1)$ all possible linear or nonlinear, local or nonlocal, Hamiltonian or nonhamiltonian, relativistic or gravitational, exterior and interior, classical and operator systems.

3.G: Mathematical and physical applications. Lie's theory is known to be at the foundation of virtually all branches of mathematics. The existence of intriguing and novel applications in mathematics originating from the Lie-Santilli theory is then self-evident.

With the understanding that mathematical studies are at their first infancy, the isotopies have already identified new branches of mathematics besides isoalgebras, isogroups and isorepresentations. We here mention: the new branch of number theory dealing with isonumbers; the new branch of functional isoanalysis dealing with T-operator special isofunctions, isotransforms and isodistribution; the new branch of topology dealing with the peculiar integro-differential topology of the isotopic theory; the new branch of the theory of manifold dealing with isomanifolds and their intriguing properties; and so on. It is hoped that interested mathematicians will contributed to these novel mathematical advances which have been identified and developed until now solely by physicists.

Lie's theory in its traditional linear-local-canonical formulation is also known to be at the foundation of all branches of contemporary physics. Profound physical implications due to the covering, nonlinear-nonlocal-noncanonical Lie-Santilli theory cannot therefore be dismissed in a credible way.

With the understanding that these latter applications too are at the beginning and so much remains to be done, let us recall the following applications of the Poincaré=-Santilli isosymmetry $\hat{P}(3.1)$ (see [61] and [62] for details):

(1) The universal invariance of all possible *conventional* gravitation [51].

(2) The geometric unification of the special and general relativities. In fact, the abstract isotope $\hat{P}(3.1)$ unifies the isosymmetry with gravitational isounit $\hat{1} = [T(x)]^{-1}$, $g(x) = T(x)\eta$, and the realization with isounit $I = diag. (1, 1, 1, 1)$ characterizing the special relativity [51]

(3) The universal invariance for all possible interior extensions of relativistic and gravitational theories [51].

(4) Reconstruction at the isotopic level of the *exact* SU(2)-isospin symmetry *under electromagnetic and weak interactions* via the use of the standard isopauli matrices (3.52) with $\lambda^2 = m_p/m_n$ [63];

(5) Quantitative representation of Rauch's interferometric measures on the 4π-spinorial symmetry via the isotopies of Dirac's equation invariant under $\hat{\mathcal{P}}(3.1)$ [69];

(6) First numerical representation of the total magnetic moment of few-body nuclei via the $\hat{SO}(3)$ symmetry and its direct representation of the deformation of the charge distribution of nucleons and consequential alteration of their intrinsic magnetic moments [69];

(7) Nonlocal representation of the Bose-Einstein correlation from first isotopic principles in full numerical agreement with the data from the UA1 experiments, while permitting a causal description of nonlocal interactions and the reconstruction of their exact Poincaré symmetry at the isotopic level ([58], [8]);

(8) Quantitative representation of the electron pairing in superconductivity [1];

(6) Quantitative-numerical representation of the behaviour of the meanlives of unstable hadrons with speed (which, as well known, are anomalous between 30 and 100 GeV and conventional between 100 and 400 GeV for the K°-system) via the isominkowskian geometrization of the physical medium in their interior ([6], [7]);

(9) Application to quarks theories via Klimyk rule for the standard isorepresentations of $\hat{SU}(3)$ with conventional quantum numbers with exact confinement of quarks (permitted by the incoherence of the interior isohilbert and exterior Hilbert spaces), and other intriguing possibilities, such as the regaining of *convergent* perturbative series for *strong* interactions (which is possible whenever $|T| \ll 1$) [68];

(10) Numerical representation of Arp's measures on quasars redshift as being due to the decrease of speed of light in chromospheres and its isominkowskian geometrization [37];

(11) Numerical representation of the joint redshift and blueshifts of pairs of quasars, particularly when proved via gamma spectroscopy to be physically connected to the associated galaxies, and prediction of a measurable isominkowskian redshift for sunlight at sunset [67];

(12) Application to local realism via the proof that Bell's inequality, von Neumann's theorem and all that are inapplicable (rather than "violated") under isotopies (evidently because of the nonunitary structure of the lifting), thus permitting an isotopic completion of quantum mechanics much along the celebrated E-P-R argument [65];

(13) Application to q-deformations, discrete time theories and other ongoing studies via their axiomatization into a form invariant under their own

time evolution and which coincide with the conventional quantum mechanical axiomatization at the abstract level [33]; and other applications (see monographs [61] and [62];

(14) Novel possibilities in theoretical biology, such as a quantitative representation of the growth of sea shells which, according to computer simulations, crack during their growth is subjected to the conventional Minkowskian geometry, while admit a normal growth under the covering isominkowskian geometry of Class III (the latter one being needed to represent bifurcations which require inversions of time) [60].

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