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Isotopic lifting of Newtonian mechanics

Ruggero Maria Santilli

Division of Mathematics, Istituto per la Ricerca di Base
Castello Principe Pignatelli, I-86075 Monteroduni (IS), Molise, Italy
e-mail: ibrrms@pinet.aip.org

Abstract

We present a simple isotopic generalization of the ordinary differential calculus, here called *isodifferential calculus*, which is based on an axiom-preserving generalization of the unit with compatible generalizations of fields, vector spaces and manifolds. The new calculus is applied to the isotopic lifting of Newton's equations of motion. We show that the isotopic equations possess capabilities which are absent for the conventional equations, such as: the representation of the actual nonspherical and deformable shape of particles; the admission of nonlocal-integral forces; and the representation of nonpotential (variationally non-self-adjoint) interactions via the unit of the theory.

Key words: Isotopies, isounits, isofields, isospaces, isoderivatives.

Levantamiento isotópico de la mecánica de Newton

Resumen

Se presenta una generalización isotópica simple del cálculo diferencial ordinario, aquí llamado *cálculo isodiferencial*, el cual está basado en la generalización preservadora del axioma de la unidad con generalizaciones compatibles de campos, espacios vectoriales y variedades. El nuevo cálculo es aplicado al levantamiento isotópico de las ecuaciones newtonianas de movimiento. Se demuestra que las ecuaciones isotópicas poseen capacidades que no están presentes en las ecuaciones convencionales, tales como: la representación de la forma no esférica y deformable real de las partículas, la admisión de fuerzas integrales no locales, y la representación de interacciones sin potencia (variacionalmente no auto-adjuntas) mediante la unidad de la teoría.

Palabras claves: Isotopías, isounidades, isocampos, isoespacios, isoderivadas,

1. Background notions on isotopies.

The basic notion of this paper, that of *isotopies*, is rather old. As Bruck [4] recalls, the notion can be traced back to the early stages of set theory where two Latin squares were said to be *isotopically related* when they can be made to coincide via permutations. Since Latin square can be interpreted as the multiplication table of quasigroups, the isotopies propagated to quasigroups and then to Jordan algebras (see, e.g., McCrimmon [11]). While at the Department of Mathematics of Harvard University in the late 1970's, Santilli [14] iniziated comprehensive studies on the isotopies of fields, vector spaces, Lie's theory and other methods. An exhaustive literature on isotopies up to 1984 can be found in bibliography [3] while subsequent references can be found in the recent monograph by Lõhmus, Paal and Sorgsepp [10].

This paper is written by a physicist to stimulate rigorous mathematical studies on the isotopies of differential calculus, here called *isodifferential calculus*. These mathematical studies are warranted because the new calculus implies simple, yet unambiguous and intriguing isotopies of contemporary analytic, algebraic and geometric theories which have lately seen a variety of novel applications in nuclear physics, particle physics, astrophysics, superconductivity and other fields [19,20]. Moreover, the recent advances in isotopies have occurred in the physical literature and they do not appear to have propagated until now to the mathematical literature.

This first paper is devoted to the presentation of the isocalculus, which is done here apparently for the first time, although the generalized calculus is implicit in other studies by this author [19,20], as we shall indicated later on. In this first paper we also recall only those aspects of the isotopies which are essential for rendering the study selfsufficient. The second paper in this series is devoted to the isotopies of classical and quantum mechanics, while the third paper presents the isotopies of the underlying geometries. Due to the emphasis on applications, our treatment is local, while abstract, realization-free profiles are merely indicated.

We should mention for completeness that the isotopies are particular cases of the so-called *genotopies* introduced by Santilli [14], which are nonlinear, nonlocal and nonhamiltonian maps, this time, violating the original axioms in favor of covering axioms (i.e., more general axioms admitting of the original ones as particular cases). In turn, the genotopies themselves are particular cases of the still broader multivalued hypergeneralizations (see, e.g., Vougiouklis' recent monograph [24]). These more general formulations are contemplated for study in subsequent papers.

The main idea of the isotopies studied by this author [14,16] is the lifting of the trivial N-dimensional unit $I = \mathrm{diag.}(1, 1, ..., 1)$ of a conventional theory into a nowhere singular, symmetric, real-valued, positive-definite and N-dimensional matrix $\hat{1} = (\hat{1}_i^j) = (\hat{1}_j^i) = \hat{T}^{-1} = (\hat{T}_i^j)^{-1} = (T_j^i)^{-1}$, i,j 1, 2, ..., N, whose elements have a smooth but otherwise arbitrary functional dependence on the local coordinates x, their derivatives $\dot{x}$, $\ddot{x}$, ..., with respect to an independent variable t and any needed additional local quantity,

$$I \rightarrow \hat{1}(x, \dot{x}, \ddot{x}, ...)\,. \tag{1.1}$$

The original theory is then reconstructed in such a way to admit $\hat{1}$ as the new left and right unit. This requires for consistency the lifting of the totality of the mathematical structure of the original theory, including fields, metric spaces, functional analysis, algebras, groups, geometries, etc. Since the new and old structures are indistinguishable at the abstract, realization-free level by construction, the lifting is a particular form of isotopy.

The fundamental isotopies are those of fields. Let $F = F(a,+,\times)$ be a field (hereon assumed to have characteristic zero) with elements a, b, ..., sum a + b, multiplication $a\times b := ab$, additive unit 0, multiplicative unit I, and familiar properties $a + 0 = 0 + a = a$, $a\times I = I\times a = a$, $\forall a \in F$, and others. We have in particular: the field $R(n,+,\times)$ of real numbers n, the field $C(c,+,\times)$ of complex numbers c, and the field $Q(q,+,\times)$ of quaternions q.

Definition 1 [18]

An "isofield" $\hat{F} = \hat{F}(\hat{a},+,\hat{\times})$ *is a ring with elements* $\hat{a} = a\times\hat{1}$, *called "isonumbers", where* $a \in F$, *and* $\hat{1}$ *is a positive-definite element generally outside* F, *equipped with two operations* (+, $\hat{\times}$), *where + is the conventional sum of* F *with conventional additive unit* 0, *and* $\hat{\times}$ *is a new multiplication*

$$\hat{a}\,\hat{\times}\,\hat{b} := \hat{a}\times\hat{T}\times\hat{b}\,,\quad \hat{1} = \hat{T}^{-1} \tag{1.2}$$

called "isomultiplication", which is such that $\hat{1}$ *is the left and right unit of* $\hat{F}$,

$$\hat{1}\,\hat{\times}\,\hat{a} = \hat{a}\,\hat{\times}\,\hat{1} \equiv \hat{a}\,,\ \forall\ \hat{a} \in \hat{F}, \tag{1.3}$$

called "isounit". Under these assumptions $\hat{F}$ *is a field, i.e., it satisfies all properties of* F *in their isotopic form for all* $\hat{a}, \hat{b}, \hat{c} \in \hat{F}$:

1. *The set* $\hat{F}$ *is closed under addition,* $\hat{a} + \hat{b} \in \hat{F}$,
2. *The addition is commutative,* $\hat{a} + \hat{b} = \hat{b} + \hat{a}$,
3. *The addition is associative,* $\hat{a} + (\hat{b} + \hat{c}) = (\hat{a} + \hat{b}) + \hat{c}$,
4. *There is an element* 0, *called t'additive unit", such that* $\hat{a} + 0 = 0 + \hat{a} = \hat{a}$,
5. *For each element* $\hat{a} \in \hat{F}$, *there is an element* $-\hat{a} \in \hat{F}$, *called the "opposite of* $\hat{a}$", *which is such that* $\hat{a} + (-\hat{a}) = 0$;
6. *The set* $\hat{F}$ *is closed under isomultiplication,* $\hat{a}\hat{\times}\hat{b} \in \hat{F}$.
7. *The multiplication is generally non-isocommutive,* $\hat{a}\hat{\times}\hat{b} \neq \hat{b}\hat{\times}\hat{a}$, *but 'isoassociative",* $\hat{a}\hat{\times}(\hat{b}\hat{\times}\hat{c}) = (\hat{a}\hat{\times}\hat{b})\hat{\times}\hat{c}$,

8. *The quantity $\hat{1}$ in the factorization $\hat{a} = a \times \hat{1}$ is the "multiplicative isounit" of $\hat{F}$ as per Eq.s (1.3)*
9. *For each element $\hat{a} \in \hat{F}$, there is an element $\hat{a}^{-\hat{1}} \in \hat{F}$, called the "isoinverse", which is such that $\hat{a}\hat{\times}(\hat{a}^{-\hat{1}}) = (\hat{a}^{-\hat{1}})\hat{\times}\hat{a} = \hat{1}$.*
10. *The set $\hat{F}$ is closed under joint isomultiplication and addition,*

$$\hat{a}\hat{\times}(\hat{b} + \hat{c}) \in \hat{F}, \quad (\hat{a} + \hat{b})\hat{\times}\hat{c} \in \hat{F}, \quad (1.4)$$

11. *All elements $\hat{a}, \hat{b}, \hat{c} \in \hat{F}$ verify the right and left "isodistributive laws"*

$$\hat{a}\hat{\times}(\hat{b}+\hat{c}) = \hat{a}\hat{\times}\hat{b} + \hat{a}\hat{\times}\hat{c}, (\hat{a}+\hat{b})\hat{\times}\hat{c} = \hat{a}\hat{\times}\hat{c} + \hat{b}\hat{\times}\hat{c}, \quad (1.5)$$

When there exists a least positive isointeger $\hat{p}$ such that the equation $\hat{p}\hat{\times}\hat{a} = 0$ admits solution for all elements $\hat{a} \in \hat{F}$, then $\hat{F}$ is said to have "isocharacteristic p". Otherwise, $\hat{F}$ is said to have "isocharacteristic zero".

We therefore have the isofield $\hat{R}(\hat{n},+,\hat{\times})$ of *isoreal numbers* $\hat{n}$; the isofield $\hat{C}(\hat{c},+,\hat{\times})$ of *isocomplex isonumbers* $\hat{c}$; and the isofield $\hat{Q}(\hat{q},+,\hat{\times})$ of *isoquaternions* $\hat{q}$ (see [18] for the *isooctonions*). Since $\hat{F}$ preserves by construction all axioms of F, it is called an *isotope* of F and the lifting $F \rightarrow \hat{F}$ is called an *isotopy*. All conventional operations dependent on the multiplication on F are generalized on $\hat{F}(\hat{a},+,\hat{\times})$, thus yielding isotopies of powers, quotients, square roots, etc. These isotopic operations are however such that $\hat{1}$ preserves all the original axiomatic properties of I, i.e., $\hat{1}^{\hat{n}} = \hat{1}*\hat{1}*...*\hat{1}$ (n-times) $= \hat{1}$, $\hat{1}^{\hat{1}} = \hat{1}$, $\hat{1}/\hat{1} = \hat{1}$, etc. (see [18] for details).

Note that the isotopy is restricted to the sum, as indicated by the symbol $\hat{F}(\hat{a},+,\hat{\times})$, because the lifting of a field into the form $\hat{F}(\hat{a},\hat{+},\hat{\times})$ inclusive of the lifting of the sum, such as $+ \rightarrow \hat{+} = + K +$ with corresponding lifting of the additive unit $0 \rightarrow \hat{0} = -K$, $K > 0$, $K \in F$, generally implies the loss of the original axioms, such as the loss of closure (1.4). Therefore, the lifting of the sum is not an isotopy. Moreover, quantities which are conventionally finite on $F(a,+,\times)$ as well as on $\hat{F}(\hat{a},+,\hat{\times})$, such as the exponentiation on F, $e^a = I + a/1! + a^2/2! + ...$, or that on $\hat{F}$, $\hat{e}^a = \hat{1} + a/1! + a\hat{\times}a/2! + ... = \{e^{a\times\hat{T}}\}\hat{1} = \hat{1}\{e^{\hat{T}\times a}\}$, become divergent under the liftings $+ \rightarrow \hat{+} = + K +$, $0 \rightarrow \hat{0} = -K$, $K \in F$ [18]. For this reason only the isotopies of the multiplication are used in applications at this writing [20].

Despite its simplicity, the lifting $F \rightarrow \hat{F}$ has significant implications in number theory itself. For instance, real numbers which are conventionally prime (under the tacit assumption of the unit I) are not necessarily prime with respect to a different unit [18]. This illustrates that most of the properties and theorems of the contemporary number theory are dependent on the assumed unit and, as such, admit intriguing isotopies. Also, the isotopies permit the conception of a new generation of cryptograms which are expected to be difficult to break because of the availability of an infinite number of different units which are not admitted by the conventional number theory.

The notion of isonumbers was presented, apparently for the first time, by this author at the conference *Differential Geometric Methods in Mathematical Physics*, held at the University of Clausthal, Germany, in 1980. The first mathematical treatment appeared in ref. [12] of 1982. A systematic study is available in above quoted ref. [18], while additional studies and applications are presented in monographs [19,20].

The mathematical and physically most important implication of isofields is that they imply, for evident consistency, corresponding isotopies of *all* quantities defined over conventional fields. Let $E(x,\delta,R)$ be an N-dimensional Euclidean space, with local chart $x = (x^k)$, $k = 1, 2, ..., N$, N-dimensional metric $\delta = \text{diag.}(1, 1, ..., 1)$ and invariant separation between two points $x, y \in E$,

$$(x - y)^2 := (x^i - y^i)\,\delta_{ij}\,(x^j - y^j) \in R(n,+,\times), \quad (1.7)$$

over the reals $R(n,+,\times)$, where the convention on the sum of repeated indices is assumed hereon.

Definition 2 [16]

An "isoeuclidean space" $\hat{E}(\hat{x},\hat{\delta},\hat{R})$ is an N-dimensional metric space defined over an isoreal isofield $\hat{R}(\hat{n},+,\hat{\times})$ with an N×N-dimensional isounit $\hat{1}$, equipped with the "isometric"

$$\hat{\delta} = (\hat{\delta}_{ij}) = \hat{T}\times\delta, \quad \hat{1} = \hat{T}^{-1} \quad (1.8)$$

where δ is the conventional Euclidean metric, local chart in contravariant and covariant forms

$$\hat{x} = \{ \hat{x}^k \} = \{ x^k \}, \qquad \hat{x}_k = \hat{\delta}_{kl} \hat{x}^l = \hat{T}_k{}^r \delta_{rl} x^l , \quad x^k \in E ; \tag{1.9}$$

and "isoseparation" among two points $\hat{x}, \hat{y} \in \hat{E}$

$$(\hat{x} - \hat{y})^{\hat{2}} := [(\hat{x}^i - \hat{y}^i) \hat{\delta}_{ij} (\hat{x}^j - \hat{y}^j)] \hat{1} \in \hat{R} \tag{1.10}$$

The "isoeuclidean geometry" is the geometry of the isoeuclidean spaces.

The primary property of the lifting $E(x,\delta,R) \rightarrow \hat{E}(\hat{x},\hat{\delta},\hat{R})$ is, again, the preservation of all original geometric axioms, thus characterizing an isotopy. In actuality, $E(x,\delta,R)$ and $\hat{E}(\hat{x},\hat{\delta},\hat{R})$ coincide at the abstract level by construction for all positive-definite isounits $\hat{1}$ (but not so for isounits of different topology [26]). This is due to the construction of the isospaces via the deformation of the metric δ into the isometric $\hat{\delta} = \hat{T}\times\delta$ while jointly the original unit I is deformed in the amount *inverse* of the deformation of δ, $\hat{1} = \hat{T}^{-1}$. This mechanism then ensures the preservation of all original geometric properties. In particular, since the original space E is flat (with respect to the unit I), the corresponding isospace $\hat{E}$ is *isoflat* that is, it verifies the axiom of flatness in isospace (with respect to the isounit $\hat{1}$). Similar results are obtained for the isotopies of the Minkowski, Riemannian, Finslerian and other spaces (see [19] for brevity). In particular, an originally curved space remains curved under isotopies.

Note that the coordinates of E and $\hat{E}$ coincide in their contravariant form, $\hat{x}^k = x^k$, but not in their covariant form, $\hat{x}_k \neq x_k$. Because of the latter occurrence, the symbol x will be used for the coordinates of conventional spaces, while the symbol $\hat{x}$ will be used for the coordinates of isospaces. When writing $\hat{\delta}(x, \ldots)$ we refer to the *projection* of the isometric $\hat{\delta}$ in the original space.

Despite its simplicity, the lifting $E(x,\delta,R) \rightarrow \hat{E}(\hat{x},\hat{\delta},\hat{R})$ also has significant implications. In fact, the functional dependence of the isounit $\hat{1}$ remains unrestricted under isotopies. The isometric $\hat{\delta}$ can therefore depend on the local coordinates x as well as their derivatives $\dot{x}, \ddot{x}, \ldots$ and any needed additional variable, $\hat{\delta} = \hat{\delta}(x, \dot{x}, \ddot{x}, \ldots)$. The isoseparation (1.10) is therefore the most general possible integro-differential separation with signature (+, +, +).

The above occurrence permits the geometrically nontrivial result according to which *a metric space can be flat under an arbitrary functional dependence of the isometric*. The understanding is that the projection of $\hat{E}(\hat{x},\hat{\delta},\hat{R})$ into the original space $E(x,\delta,R)$ is curved. Note that Riemannian metrics g(x) are a *particular case* of the broader isoeuclidean isometric $\hat{\delta}(x, \dot{x}, \ddot{x}, \ldots)$. This indicates that the N-dimensional Riemannian space $\mathfrak{R}(x,g,R)$ over the reals can be reinterpreted as the isospace $\hat{E}(\hat{x},\hat{\delta},\hat{R})$, $\hat{\delta} = g(x)$, over the isoreals via the factorization $g(x) = \hat{T}(x)\delta$. The assumption of the isounit $\hat{1} = T^{-1}$ then eliminates the Riemannian curvature in isospace with intriguing physical applications in gravitation [20].

Isogeometries have novel properties which do not appear to have propagated as yet into the mathematical literature. For instance, the conventional trigonometry on the two-dimensional Euclidean space $E(x,\delta,R)$, δ = diag. (1, 1) (Gauss plane) is lost under lifting to a two-dimensional Riemannian space $\mathfrak{R}(x,g(x),R)$, but trigonometry can be reformulated in the two-dimensional isospace $\hat{E}(\hat{x},\hat{\delta}(x,\dot{x},\ddot{x}\ldots),\hat{R})$ resulting in the so-called *isotrigonometry* (see [19], App. 6.A, for brevity). An intriguing application is the formulation of the *Pythagorean theorem for a triangle with curved sides* (because for each given such triangle, there exists an isotopy such that its image in isospace is an ordinary triangle with rectilinear sides).

Similarly, all infinitely possible spheroidal ellipsoids in three-dimensional Euclidean space $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1 \in R(n,+,\times)$, a, b, c, $\neq 0$, are unified by the perfect sphere in isospace called *isosphere*

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right) \hat{1} = \hat{1} \in \hat{R}(\hat{n},\hat{+},\hat{\times}) , \tag{1.11a}$$

$$\hat{1} = \text{diag.} (a^2 , b^2, c^2) , \hat{T} = \text{diag.} (a^{-2}, b^{-2}, c^{-2}) . \tag{1.11b}$$

In fact, under isotopies the semiaxes (1, 1, 1) of the original perfect sphere are deformed into the values (a^2, b^2, c^2), but the corresponding units (+1, +1, +1) are deformed of the *inverse* amounts (a^{-2}, b^{-2}, c^{-2}), thus preserving the perfect spheridicity in isospace. When the conditions of positive-definiteness and non-singularity of the isounit are relaxed, the isosphere unifies all possible compact and noncompact quadrics and cones in three-dimension. The use of yet more general isounits then yields new geometric notions, such as an isosphere whose isounit is singular or a distribution.

The notions of isoeuclidean, isominkowskian and isoriemannian spaces over isofields were introduced by this author in paper [16] of 1983, studied in various memoirs, and presented in a comprehensive way in monograph [19], including the isotopies of the Minkowskian, Riemannian and other geometries.

The notion of *isocontinuity* on an isospace was first studied by Kadeisvili [8] and resulted to be easily reducible to that of conventional continuity because the *isomodulus* $\uparrow \hat{f}(\hat{x}) \uparrow$ of a function $\hat{f}(\hat{x})$ on $\hat{E}(\hat{x},\hat{\delta},\hat{R})$ over $\hat{R}(\hat{n},+,\hat{\times})$ is given by the conventional modulus $|\hat{f}(\hat{x})|$ multiplied by the positive-definite isounit $\hat{1}$,

$$\uparrow \hat{f}(\hat{x}) \uparrow \; = |\hat{f}(\hat{x})| \times \hat{1} > 0 . \quad (1.12)$$

As an illustration, an infinite sequence $\hat{f}_1, \hat{f}_2, \ldots$ is said to be *strongly isoconvergent* to $\hat{f}$ when

$$\mathrm{Lim}_{k \to \infty} \uparrow \hat{f}_k - \hat{f} \uparrow \; = 0 , \quad (1.13)$$

while the *isocauchy condition* can then be expressed by

$$\uparrow \hat{f}_m - \hat{f}_n \uparrow \; < \hat{\delta} = \delta \times \hat{1} , \quad (1.14)$$

where δ is real and m and n are greater than a suitably chosen $N(\delta)$. The isotopies of other notions of continuity, limits, series, etc. can be easily constructed [26]. Note that functions which are conventionally continuous are also isocontinuous. Similarly, a series which is strongly convergent is also strongly isoconvergent.

However, *a series which is strongly isoconvergent is not necessarily strongly convergent* (ref. [19], p. 271). As a result, *a series which is conventionally divergent can be turned into a convergent form under a suitable isotopy.* This mathematically trivial property has rather important physical applications, e.g., for the first construction of a theory of strong interactions with convergent perturbative expansions [27].

The notion of an N-dimensional *isomanifold* was first studied by Tsagas and Sourlas [23,24]. In this paper we use the following simplest possible realization of isomanifolds. Since an N×N-dimensional isounit is positive-definite, it can always be diagonalized into the form

$$\hat{1} = \mathrm{diag.}\,(b_1^{-2}, b_2^{-2}, \ldots, b_N^{-2}) > 0 , \quad b_k > 0 , \quad (1.15)$$

Consider then N isoreal isofields $\hat{R}_k(\hat{n},+,\hat{\times})$ each characterized by the isounit $\hat{1}_k = b_k^{-2}$ with (ordered) Cartesian product

$$\hat{R}^N = \hat{R}_1 \times \hat{R}_2 \times \ldots \times \hat{R}_N . \quad (1.16)$$

Since $\hat{R}_k \approx R$, it is evident that $\hat{R}^N \approx R^N$, where R^N is the Cartesian product of N conventional fields $R(n,+,\times)$. But the total unit of $\hat{R}^N$ is expression (1.15). Therefore, one can introduce a topology on $\hat{R}^N$ via the simple isotopy of the conventional topology on R^N,

$$\hat{\tau} = \{ \emptyset , \hat{R}^N, \hat{B}_i \} , \quad (1.17)$$

where $\hat{B}_i$ represents the subset of $\hat{R}^n$ defined by

$$\hat{B}_i = \{ \hat{P} = (\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) \,?\, \hat{n}_i < \hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n <$$

$$< \hat{m}_i, \hat{n}_i , \hat{m}_i , a_k \in \hat{R} \} . \quad (1.18)$$

As one can see, the above topology coincides everywhere with the conventional topology τ of R^n *except at the isounit* $\hat{1}$. In particular, $\hat{\tau}$ is everywhere local-differential, except at $\hat{1}$ which can incorporate integral terms. In this sense $\hat{\tau}$ is called an *isotopology* or an *integro-differential topology.*

Definition 3 [23]

A "topological isospace" $\hat{\tau}(\hat{R}^N)$ is the isospace $\hat{R}^N$ equipped with the isotopology $\hat{\tau}$. A "Cartesian isomanifold" $\hat{M}(\hat{R}^N)$ is the topological isospace $\hat{\tau}(\hat{R}^N)$ equipped with a vector structure, an affine structure and the mapping

$$\hat{f} : \hat{R}^n \to \hat{R}^n , \quad \hat{f} : \hat{a} \to \hat{f}(\hat{a}) = \hat{a}, \;\; \forall\, \hat{a} \in \hat{R} . \quad (1.18)$$

An "isoeuclidean isomanifold" $\hat{M}(\hat{E}(\hat{x},\hat{\delta},\hat{R}))$ occurs when the N-dimensional isospace $\hat{E}$ is realized as the Cartesian product

$$\hat{E}(x,\hat{\delta},\hat{R}) \approx \hat{R}_1 \times \hat{R}_2 \times \ldots \times \hat{R}_N , \quad (1.19)$$

and equipped with the isotopology $\hat{\tau}$ with isounit (1.15).

The extension of the above definition to nondiagonal isounits $\hat{1}$ can be trivially achieved, e.g., by assuming that the individual isounits $\hat{1}_k$ are positive-definite N×N-dimensional nondiagonal matrices such to yield the assumed total unit $\hat{1}$ via the ordered Cartesian product

$$\hat{1} = \hat{1}_1 \times \hat{1}_2 \times \ \ldots \ \times \hat{1}_N . \qquad (1.20)$$

For all additional aspects of isomanifolds and related topological properties we refer the interested reader to Tsagas and Sourlas [23,24]. It should be noted that their study is referred to $M(\hat{R}^N)$, rather than to $\hat{M}(\hat{R}^N)$ because of the use of the *conventional* topology τ (i.e. a topology with the conventional N×N-dimensional unit I). The extension to $\hat{M}(\hat{E})$ with the isotopology $\hat{\tau}$ is introduced here apparently for the first time.

The above notions on isotopies are sufficient for the limited objectives of this paper. In regard to additional isotopies, we merely mention that the *Lie-isotopic theory* submitted by this author [14,16,19 and today called *Lie-Santilli isotheory* (see monographs [2,7,10,21] or review paper [9] and literature quoted therein). In essence, Lie's theory in its contemporary formulation (on conventional spaces over conventional fields) is linear, local and canonical and, as such, it possesses limitations in its applications. The isotopies of Lie's theory are the most general possible nonlinear, nonlocal and noncanonical maps capable of reconstructing linearity, locality and canonicity when formulated in isospaces over isofields. As such, the isotopies imply a considerable broadening of the applications of the conventional Lie theory while preserving its axioms at the abstract level.

The isotopies of functional analysis, called *isofunctional analysis*, were introduced by Kadeisvili [8], who also introduced a classification of isounits into five topologically different classes, today called *Kadeisvili's classification*. This paper is devoted to the isotopies of Kadeisvili's Class I, i.e., those with a well behaved and *positive-definite* isounits, the isotopies of Class II occur when the isounits satisfy the same conditions except that they are *negative-definite*; the isotopies of Class III are the union of Classes I and II; those of Class IV include all preceding Classes plus *singular* isounits; and those of Class V include all preceding ones plus isounits of unrestricted characteristics, such as step-functions, distributions, lattices, etc.

Kadeisvili's classification is significant because it illustrates the broad character of the isotopies. For instance, Lie's theory is unique (because referred to the single unit I), while the Lie-Santilli isotheory admits five topologically distinct classes (because based on five distinct isounits). It should be stressed that, despite all the studies conducted to date, the isotopies remain vastly unexplored at this writing. In fact, only the isotopies of Class I, II and III have been preliminarily studied until now, while those of Classes IV and V have remained essentially unexplored.

Note that this paper is formulated for isotopies of Class I, but its content can be readily extended to those of Classes II and III, although the extension to Classes IV and V require specific studies.

EXAMPLES. Some specific examples of isounits used in ref. [20] for various applications may be of assistance for mathematicians in understanding the physical needs and identifying the ensuing mathematical requirements. One of the simplest possible applications of the isotopies is the representation of *nonspherical* shapes of particles and their *deformations* due to external forces or collisions. For the simplest possible case of spheroidal ellipsoids in three dimension, the isounit is given by

$$\hat{1} = \text{diag.}\,(n_1^{-2} , n_2^{-2}, n_3^{-2}) , \qquad (1.21)$$

where the semiaxes n_k^2 are sufficiently well behaved, real valued and positive-definite functions of local quantities, such as the intensity of external fields, the local pressure, etc.

The next simplest possible example is the representation of systems which are open-nonconservative because of exchanges of physical quantities with an external system. In this case the isounit is a well behaved function of local quantities admitting of the value I as a particular case, e.g.,

$$\hat{1} = e^{f(t,\, x,\, \dot{x},\, ..)} , \quad \hat{1}_{|f=0} = I . \qquad (1.22)$$

Isounits of this type permit the representation of continuously decaying angular momenta; particles moving within resistive media under nonhamiltonian but local-differential forces (see later on); the growth of sea shells; and other nonconservative systems.

The next class of isounits needed in applications is of nonlocal-integral type, that is, dependent on an integral over a surface or a volume. An illustration is given by the two electrons of the Cooper pair in superconductivity which experience an *attractive* interaction against their *repulsive* Coulomb force. The use of the quantum mechanical Coulomb law with conventional unit $I = \text{diag.}(1, 1, 1)$ leads to repulsion. On the contrary, Animalu [1] has shown that the following isounit

$$\hat{1} = e^{\int d^3x\, \phi_\uparrow^\dagger(x)\, \phi_\downarrow(x)} \text{diag.}(1, 1, 1), \qquad (1.23)$$

permits a quantitative interpretation of the attraction among the two identical electrons in a way conform with experimental evidence, where $\phi_\uparrow$ and $\phi_\downarrow$ are the wavefunctions of the two electrons with related spin orientation $\uparrow$ and $\downarrow$. The exponent of (1.23) then illustrates the type of nonlocality needed for applications. Note that when the overlapping of the two wavepackets $\phi_\uparrow$ and $\phi_\downarrow$ is no longer appreciable, the integral in the exponent of isounit (1.23) is ignorable and $\hat{1}$ recovers the conventional unit I.

In general, the isounit used in application is a positive-define matrix with the dimension of the used carrier space (two-, three- and four-dimension for problems in the plane, space and space-time, respectively) which is generally *nondiagonal* whose elements have a local-differential as well as nonlocal integral dependence on local physical quantities.

Remark. In the conventional Euclidean space $E(x,\delta,R)$ the unit of the field R, which is the trivial *number* +1, is different than the unit of the space, which is the unit *matrix* $I = \text{diag.}(1, 1, 1)$, although the field could be trivially reformulated to admit the latter unit. Under isotopies *the isounit of all mathematical structures must be the same.* Therefore, in the isoeuclidean spaces $\hat{E}(\hat{x},\hat{\delta},\hat{R})$ the isounit $\hat{1}$ of the isofield $\hat{R}$ *coincides* with the isounit of the isospace.

2. Isodifferential calculus on isomanifolds.

Let $E(x,\delta,R)$ be the ordinary N-dimensional Euclidean space with local coordinates $x = \{x^k\}$, $k = 1, 2, ..., N$, and metric $\delta = \text{diag.}(1, 1, 1)$ over the reals $R(n,+,\times)$. Let $\hat{E}(\hat{x},\hat{\delta},\hat{R})$ be its isotopic image with local coordinates $\hat{x} = \{\hat{x}^k\}$ and isometric $\hat{\delta} = \hat{T}\delta$ over the isoreals $\hat{R}(\hat{n},+,\hat{\times})$. Let the isounit be given by the N×N nowhere singular, symmetric, real-valued and positive-definite matrix $\hat{1} = (\hat{1}_i{}^j) = (1_i{}^j) = \hat{T}^{-1} = (T_i{}^j)^{-1} = (T_i{}^j)^{-1}$ whose elements have a smooth but otherwise arbitrary functional dependence on the local coordinates, their derivatives with respect to an independent variable and any needed additional quantity, $\hat{1} = \hat{1}(\hat{x},...)$. The following properties then hold from Definition 2:

$$\hat{x}^k \equiv x^k, \qquad \hat{x}_k = \hat{\delta}_{ki}\hat{x}^i = \hat{T}_k{}^i\delta_{ij}\hat{x}^j = \hat{T}_k{}^i\delta_{ij}x^j =$$

$$= \hat{T}_k{}^i x_i, \quad x_i = \delta_{ij}x^j, \qquad (2.1a)$$

$$\hat{x}^i\hat{\delta}_{ij}\hat{x}^j = \hat{x}^i\hat{T}_i{}^j\delta_{jm}\hat{x}^m = \hat{x}_i\hat{\delta}^{ij}\hat{x}_j \equiv \hat{x}^k\hat{x}_k = \hat{x}_k\hat{x}^k,$$

$$\hat{\delta}^{ij} = [(\hat{\delta}_{mn})^{-1}]^{ij}, \qquad (2.1b)$$

$$x^i\delta_{ij}x^j = x_i\delta^{ij}x_j = x^ix_i = x_ix^i, \qquad \delta^{ij} = [(\delta_{mn})^{-1}]^{ij}. \qquad (2.1c)$$

Let $\hat{M}[\hat{E}(\hat{x},\hat{\delta},\hat{R})]$ be an isomanifold on $\hat{E}$ as per Definition 3 hereon referred as $\hat{M}[\hat{E}]$. The *isodifferential calculus* on $\hat{M}(\hat{E})$ can be defined as an isotopic lifting of the conventional differential calculus on $M(E)$, that is, a lifting based on the generalization of the unit I of $M(E)$ into the isounit $\hat{1}$ of $\hat{M}(\hat{E})$, under the condition of preserving the original axioms and properties of the ordinary differential calculus, including the condition of the invariance of the isounit (see below).

Definition 4

The "first-order isodifferentials" of the contravariant and covariant coordinates $\hat{x}^k$ and $\hat{x}_k$, on $\hat{M}(\hat{E})$ are given by

$$\hat{d}\hat{x}^k = \hat{1}^k{}_i(x, ...)\,dx^i, \quad \hat{d}\hat{x}_k = \hat{T}_k{}^i(x, ...)\,dx_i, \qquad (2.2)$$

where the expressions $\hat{d}\hat{x}^k$ and $\hat{d}\hat{x}_k$ are defined on $\hat{M}(\hat{E})$ while the corresponding expressions $\hat{1}^k{}_i dx^i$ and $\hat{T}_k{}^i dx_i$ are the projections on $M(E)$. Let $\hat{f}(\hat{x})$ be a sufficiently smooth isofunction on a closed domain $\hat{D}(\hat{x}^k)$ of contravariant coordinates $\hat{x}^k$ on $\hat{M}(\hat{E})$. Then the "isoderivative" at a point $\hat{a}^k \in \hat{D}(\hat{x}^k)$ is given by

$$\hat{f}'(\hat{a}^k) = \frac{\hat{\partial}\hat{f}(\hat{x})}{\hat{\partial}\hat{x}^k}\Big|_{\hat{x}^k = \hat{a}^k} = \hat{T}_k{}^i\frac{\partial f(x)}{\partial x^i}\Big|_{\hat{x}^k = \hat{a}} =$$

$$= \text{Lim}_{\hat{d}x^k \to 0^k}\frac{\hat{f}(\hat{a}^k + \hat{d}\hat{x}^k) - \hat{f}(\hat{a}^k)}{\hat{d}x^k}, \qquad (2.3)$$

where use of Kadeisvili's [12] notions of isocontinuity, isolimits and isoconvergence is assumed, $\hat{\partial}\hat{f}(\hat{x})/\hat{\partial}\hat{x}^k$ is computed on $\hat{M}(\hat{E})$ and $\hat{T}_k{}^i\partial f(x)/\partial x^i$ is the projection in

M(E). *The "isoderivative" of a smooth isofunction* $\hat{f}(\hat{x})$ *of the covariant variable* $\hat{x}_k$ *at the point* $\hat{a}_k \in \hat{D}(\hat{x}_k)$ *is given by*

$$\hat{f}'(\hat{a}_k) = \frac{\hat{\partial} \hat{f}(\hat{x})}{\hat{\partial} \hat{x}_k}|_{\hat{x}_k = \hat{a}_k} = \hat{1}^k{}_i \frac{\partial f(x)}{\partial x_i}|_{\hat{x}_k = \hat{a}_k} =$$

$$= \mathrm{Lim}_{\hat{d}\hat{x}_k \to \hat{0}_k} \frac{\hat{f}(\hat{a}_k + \hat{d}x_k) - \hat{f}(\hat{a}_k)}{\hat{d}\hat{x}_k} \qquad (2.4)$$

The above definition and the axiom-preserving character of the isotopies then permit the lifting of the various aspects of the conventional differential calculus. We here mention for brevity the following isotopies: the *isodifferentials of an isofunction* of contravariant (covariant) coordinates $\hat{x}^k$ ($\hat{x}_k$) on $\hat{E}(\hat{x},\hat{\delta},\hat{R})$ are defined via the isoderivatives according to the respective rules

$$\hat{d}\hat{f}(\hat{x})|_{contrav.} = \frac{\hat{\partial}\hat{f}}{\hat{\partial}\hat{x}^k} \hat{d}\hat{x}^k = \hat{T}_k{}^i \frac{\partial f}{\partial x^i} \hat{1}^k{}_j dx^j = df(x) \quad (2.5a)$$

$$\hat{d}\hat{f}(x)|_{covar.} = \frac{\hat{\partial}\hat{f}}{\hat{\partial}\hat{x}_k} \hat{d}\hat{x}_k = \hat{1}^k{}_i \frac{\partial f}{\partial x_i} \hat{T}_k{}^j dx_j = df(x)\,; \quad (2.5b)$$

an iteration of the notion of isoderivative leads to the *second-order isoderivatives* (no sums on k)

$$\frac{\hat{\partial}^2 \hat{f}(\hat{x})}{\hat{\partial} \hat{x}^{k\,2}} = \hat{T}_k{}^i \hat{T}_k{}^j \frac{\partial^2 f(x)}{\partial x^i \partial x^j}, \quad \frac{\hat{\partial}^2 \hat{f}(\hat{x})}{\hat{\partial} \hat{x}_k{}^2} =$$

$$= \hat{1}^k{}_i \hat{1}^k{}_j \frac{\partial^2 f(x)}{\partial x_i \partial x_j} \qquad (2.6)$$

and similarly for isoderivatives of higher order; the *isolaplacian* on $\hat{E}(\hat{x},\hat{\delta},\hat{R})$ is given by

$$\hat{\Delta} = \hat{\partial}_k \hat{\partial}^k = \hat{\partial}^i \hat{\delta}_{ij} \hat{\partial}^j = \partial^i \hat{\delta}_{ij} \partial^j = \hat{1}^i{}_k \partial^k \delta_{ij} \partial^j,$$

$$\hat{\partial}_k = \hat{\partial}/\hat{\partial}\hat{x}^k, \ \partial_k = \partial/\partial x^k, \text{ etc.}, \qquad (2.7)$$

and and results to be different than the corresponding expression on a Riemannian space $\Re(x,g,R)$ with metric $g(x) = \hat{\delta}$, $\Delta = \hat{\delta}^{-1/2} \partial_i \hat{\delta}^{1/2} \hat{\delta}^{ij} \partial_j$

A few examples are in order. First note the following properties derived from definitions (2.3) and (2.4),

$$\hat{\partial}\hat{x}^i / \hat{\partial}\hat{x}^j = \delta^i{}_j, \quad \hat{\partial}\hat{x}_i / \hat{\partial}\hat{x}_j = \delta_i{}^j, \quad \hat{\partial}\hat{x}_i / \hat{\partial}\hat{x}^j = \hat{T}_i{}^j,$$

$$\hat{\partial}\hat{x}^i / \hat{\partial}\hat{x}_j = \hat{1}^i{}_j. \qquad (2.8)$$

Next, we have the simple isoderivatives

$$\frac{\hat{\partial}(\hat{x}_k \hat{x}^k)}{\hat{\partial}\hat{x}^r} = \frac{\hat{\partial}(\hat{x}^i \hat{\delta}_{ij} \hat{x}^j)}{\hat{\partial}x^r} = \hat{T}_r{}^i \frac{\partial (x^i \delta_{ij} x^j)}{\partial x^i} = \hat{T}_r{}^i 2x^i = 2\hat{x}_r \qquad (2.9a)$$

$$\frac{\hat{\partial} \ln \hat{\psi}(\hat{x})}{\hat{\partial}\hat{x}^k} = \hat{T}_k{}^i \frac{\partial \ln \psi(x)}{\partial x^i} = \frac{\hat{1}}{\hat{\psi}(\hat{x})} \frac{\hat{\partial} \hat{\psi}(\hat{x})}{\hat{\partial} \hat{x}^k}, \qquad (2.9b)$$

and similarly for other cases.

For completeness we mention the (indefinite) *isointegration* which, when defined as the inverse of the isodifferential, is given by

$$\hat{\int} \hat{d}\hat{x} = \int \hat{T} \hat{1}\, dx = \int dx = x, \qquad (2.10)$$

namely, $\hat{\int} = \int \hat{T}$. Definite isointegrals are formulated accordingly.

The above basic notions are sufficient for our needs at this time. The class of isodifferentiable isofunctions of order m will be indicated $\hat{C}^m$. Systematic studies on the isotopies of the various theorems of the conventional calculus (see, e.g., [32]) will be studied elsewhere

Remark 1. The isodifferential, isoderivative and isodifferentiation verify the condition of preserving the basic isounit $\hat{1}$. Mathematically, this condition is *necessary* to prevent that a set of isofunctions $\hat{f}(\hat{x})$, $\hat{g}(\hat{x})$, ..., on $\hat{E}(\hat{x},\hat{\delta},\hat{R})$ over the isofield $\hat{R}(\hat{n},+,\hat{\times})$ with isounit $\hat{1}$ are mapped under isoderivative into a set of isofunctions $\hat{f}'(\hat{x})$, $\hat{g}'(\hat{x})$, ..., defined over a *different* field because of the alteration of the isounit. Physically, the condition is also *necessary* because the unit is a pre-requisite for measurements. The lack of conservation of the unit therefore implies the lack of consistent physical applications.

As an example, the following alternative definition of the isodifferential

$$\hat{d}\hat{x}^k = d(\hat{1}^k{}_i x^i) = [(\partial_i \hat{1}^k{}_r) x^r + \hat{1}^k{}_i] dx^i = \hat{W}^k{}_i dx^i, \qquad (2.11)$$

would imply the alteration of the isounit, $\hat{1} \to \hat{W} \neq \hat{1}$, thus being mathematically and physically unacceptable.

Nevertheless, when using isoderivatives on independent isomanifolds, say, isoderivatives on coordinates and time, the above rule does not apply and we have

$$\partial_t \partial_k \hat{f}(\hat{t}, \hat{x}) = \partial_t [\partial_k \hat{f}(\hat{t}, \hat{x})] = \partial_t [T_k{}^i(t, x, ...) \partial_i f(t, x)] . \quad (2.12)$$

Additional properties of the isodifferential calculus will be identified during the course of our analysis.

Remark 2. The conventional differential calculus is local-differential on M(E). The isodifferential calculus is local-differential on $\hat{M}(\hat{E})$ but, when projected on M(E), it becomes *integro-differential* because it incorporates integral terms in the isounit.

Remark 3. A representative application of the conventional differential calculus is the characterization of the equations of motion of a satellite in a stationary orbit in empty space. A representative application of the isodifferential calculus is the characterization of the equations of motion of the same satellite, this time, during re-entry in our atmosphere. In the former case the actual shape and dimension of the satellite do not affect its trajectory. Therefore, the satellite can be well approximated as a *massive point,* according to Galilei's [5] original conception, yielding a *local-differential problem.* In the latter case the actual shape and dimension of the satellite directly affects its trajectory because two satellites with all the same initial re-entry data of mass, speed, etc. but different shapes and dimensions have different re-entry trajectories. In this latter case the Galilean approximation of the satellite into a massive point is no longer applicable, and we have an *integro-differential problem,* that is, a problem with conventional local-differential center-of-mass trajectories x(t) plus additional corrective terms with the structure of surface integrals representing the contribution of the shape and dimension of the satellite. The notion of integro-differential topology has been conceived by this author [19] in an attempt to characterize the latter systems.

3. Isotopic lifting of Newtonian mechanics.

Newton's equations have remained essentially unchanged since their formulation in 1687 [13]. Their re-inspection is now warranted because classical Hamiltonian mechanics has been constructed to represent Newton's equations and, in turn, quantum mechanics has been constructed as an operator image of Hamiltonian mechanics. The applicability of these mechanics is essentially restricted to local-differential and potential systems, while the advancement of knowledge in various disciplines is requesting the treatment of nonlocal-integral and nonpotential systems. It then follows that a possible broadening of contemporary dynamics must originate from its foundations, Newton's equations.

In this section we introduce, apparently for the first time, the nonlinear, nonlocal and nonhamiltonian isotopies of Newton's equations as characterized by the isodifferential calculus. The isotopies have been selected over a variety of other possibilities because of their axiom-preserving character as well as of the consequential broadening of classical and quantum mechanics outlined in subsequent sections.

The contemporary formulation of Newton's equations requires the tensorial space $S(t,x,v) = E(t) \times E(x,\delta,R) \times E(v,\delta,R)$ where E(t) is the one-dimensional space representing time t, $E(x,\delta,R)$ is the conventional three-dimensional Euclidean space with local trajectories $x(t) = \{x^k\} = \{x, y, z\}$ and $E(v,\delta,R)$ is the tangent space TE which, at this Newtonian level, can be considered as an independent space representing the contravariant velocities $v = \{v^k\} = dx^k/dt$. Newton's equations for a test body of mass m = const. ($\neq 0$) moving within a resistive medium (e.g., our atmosphere) can then be written

$$m \, dv_k / dt - F^{SA}{}_k(t, x, v) - F^{NSA}{}_k(t, x, v) = 0 , \quad k = 1, 2, 3 \,(= x, y, z) , \quad (3.1)$$

where SA (NSA) stands for *variational self-adjointness* (*variational non-self-adjointness*), i.e.. the verification (violation) of the necessary and sufficient conditions for the existence of a potential U(t, x, v) originally due to Helmholtz [6] (see monograph [15] for historical notes and systematic studies). It should be recalled that in Newtonian mechanics the potential U(t, x, v) must be linear in the velocities to avoid a redefinition of the mass,

$$U(t, x, v) = U_k(t, x) v^k + U_o(t, x) . \quad (3.2)$$

Eq.s (3.1) can then be written

$$\left\{ m \frac{dv_k}{dt} - \frac{d}{dt} \frac{\partial U(t, x, v)}{\partial v^k} + \frac{\partial U(t, x, v)}{\partial x^k} - F^{NSA}{}_k(t, x, v) \right\}^{NSA} =$$

$$= \left\{ m \frac{dv_k}{dt} - \frac{\partial U_k(t, x)}{\partial x^s} \frac{dv^s}{dt} + \frac{\partial U_o(t, x)}{\partial x^k} - \right.$$

$$- F^{NSA}{}_k(t, x, v) \Big\}^{NSA} = 0 , \qquad (3.3)$$

namely, they are not in general derivable from Lagrange's [14] or Hamilton's [8] equations in the local chart {t, x, v}, as well known [20,21] (see later on for coordinate transforms). The extension to systems of n particles with masses m_k ($\neq 0$) is straighforward and will be ignored for brevity.

The representation space of the desired isotopic image of Newton's equations is given by the Kronecker product of isospaces $\hat{S}(\hat{t},\hat{x},\hat{v}) = \hat{E}(\hat{t})\times\hat{E}(\hat{x},\hat{\delta},\hat{R})\times\hat{E}(\hat{v},\hat{\delta},\hat{R})$ with total isounit $\hat{1}_{tot} = \hat{1}^o{}_o \times \hat{1} \times \hat{1}$, where $\hat{1}^o{}_o = (\hat{T}_o{}^o)^{-1}$ is the (one-dimensional) *time isounit*, $\hat{1} = (\hat{1}^k{}_i) = (\hat{T}_k{}^i)^{-1}$ is the (three-dimensional) *space isounit*, and the isounits of isospaces $\hat{E}(\hat{x},\hat{\delta},\hat{R})$ and $\hat{E}(\hat{v},\hat{\delta},\hat{R})$ have bee. assumed to coincide for simplicity. For clarity, we continue to differentiate the *isotime* $\hat{t}$, *isocoordinates* $\hat{x}^k(\hat{t})$ and *isovelocities* $\hat{v}^k(t)$ from the original respective quantities t, x^k and v^k, with the following relationships in addition to (2.1)

$$\hat{t} = t, \qquad \hat{v}^k \equiv v^k , \qquad \hat{v}_k = \hat{\delta}_{kj}\, \hat{v}^j = \hat{T}_k{}^i\, \delta_{ij}\, \hat{v}^j =$$

$$= \hat{T}_k{}^i\, v_i \neq v_k = \delta_{ki}\, x^i . \qquad (3.4)$$

Definition 5

The isotopic lifting of Newton's equations (3.3) *in isospace* $\hat{S}(\hat{t},\hat{x},\hat{v})$, *here called "isotopic Newton equations", are given by*

$$\hat{\Gamma}_k(\hat{t}, \hat{x}, \hat{v}) = \hat{m}\frac{\hat{d}\hat{v}_k}{\hat{d}\hat{t}} - \frac{\hat{d}}{\hat{d}\hat{t}}\frac{\partial \hat{U}(\hat{t}, \hat{x}, \hat{v})}{\partial \hat{v}^k} + \frac{\partial \hat{U}(\hat{t}, \hat{x}, \hat{v})}{\partial \hat{x}^k} =$$

$$= \hat{m}\frac{\hat{d}\hat{v}_k}{\hat{d}\hat{t}} - \frac{\partial \hat{U}_k(\hat{t}, \hat{x})}{\partial \hat{x}^i}\frac{\hat{d}\hat{x}^i}{\hat{d}\hat{t}} + \frac{\partial \hat{U}_o(\hat{t}, \hat{x})}{\partial \hat{x}^k} = 0 \qquad (3.5a)$$

$$\hat{U}(\hat{t}, \hat{x}, \hat{v}) = \hat{U}_k(\hat{t}, \hat{x})\, \hat{v}^k + \hat{U}_o(\hat{t}, \hat{x}) , \qquad (3.5b)$$

where we have used properties (2.7), $\hat{m}$ = *const* ($\neq 0$) *is the "isotopic mass", that is, the image of the Newtonian mass in isospace and one should note the preservation of the linearity of isopotential* (3.5b) *in* $\hat{v}^k$.

We are now equipped to prove the following:

Theorem 1

All possible sufficiently smooth, regular, but nonlinear, nonlocal-integral and variationally non-self-adjoint Newton's equations (3.3) *always admit in a neighborhood* $\hat{D}(\hat{S})$ *of a point* (t, x, v) *the representation in terms of the isotopic Newton's equations* (3.5)

$$\hat{m}\frac{\hat{d}\hat{v}_k}{\hat{d}\hat{t}} - \frac{\hat{d}}{\hat{d}\hat{t}}\frac{\partial \hat{U}(\hat{t}, \hat{x}, \hat{v})}{\partial \hat{v}^k} + \frac{\partial \hat{U}(\hat{t}, \hat{x}, \hat{v})}{\partial \hat{x}^k} =$$

$$= \hat{T}_k{}^i \Big\{ m\frac{dv_i}{dt} - \frac{\partial U_i(t, x)}{\partial x^s}\frac{dx^s}{dt} + \frac{\partial U_o(t, x)}{\partial x^i}$$

$$- F^{NSA}{}_i(t, x, v) \Big\} = 0 . \qquad (3.6)$$

Proof. When projected in the original space S(t, x, v), Eq.s (3.5) can be written

$$\hat{m}\hat{T}_o{}^o\frac{d(\hat{T}_k{}^i v_i)}{dt} - \hat{T}_o{}^o\frac{d}{dt}\hat{T}_k{}^i\frac{\partial \hat{U}(\hat{t}, \hat{x}, \hat{v})}{\partial \hat{v}^i} +$$

$$+ \hat{T}_k{}^i\frac{\partial \hat{U}(\hat{t}, \hat{x}, \hat{v})}{\partial x^i} =$$

$$= \hat{m}\,\hat{T}_o{}^o\,\hat{T}_k{}^i\frac{d\,v_i}{dt} - \hat{T}_o{}^o\,\hat{T}_k{}^i\frac{\partial \hat{U}_i(t, x)}{\partial x^s}v^s +$$

$$\hat{T}_k{}^i\frac{\partial \hat{U}_o(t, x)}{\partial x^i} + \hat{m}\,\hat{T}_o{}^o\frac{d\hat{T}_k{}^i}{dt}v_i = 0. \qquad (3.7)$$

Sufficient conditions for identities (3.6) are then then given by

$$\hat{m}\, T_o{}^o\, dv_i / dt = m\, dv_i / dt , \qquad (3.8a)$$

$$\hat{T}_o{}^o\frac{\partial \hat{U}_i(t, x)}{\partial x^s}v^s = \frac{\partial U_i(t, x)}{\partial x^s}v^s , \qquad (3.8b)$$

$$\frac{\partial \hat{U}_o(t, x)}{\partial x^i} = \frac{\partial U_o(t, x)}{\partial x^i}, \qquad (3.8c)$$

$$\hat{m}\,\hat{T}_o{}^o\frac{d\hat{T}_k{}^i(t, x, ...)}{dt}v_i = -\hat{T}_k{}^i\, F^{NSA}{}_i(t, x, v) . \qquad (3.8d)$$

which, under the assumed continuity and regularity conditions (see [20] for details) always admits a solution in the unknown quantities $\hat{m}$, $\hat{T}_o^o$, $\hat{T}_k^i$, $\hat{U}_k$ and $\hat{U}_o$ for given equations (3.3). In fact, system (3.8) is overdetermined and the following solution exists for *diagonal* space isounit and *constant* time isounit,

$$\hat{1}^i_k = \delta^i_k \, e^{f_k(t,\, x,\, v)}, \qquad \hat{1}^o_o = \text{constant} > 0\,, \tag{3.9}$$

for which

$$\hat{m}\, T_o^o \equiv m\,,\ \hat{U}_k(t, x) = \hat{1}^o_o\, U_k(t, x)\,,\ \hat{U}_o(t, x) = U_o(t, x)\,, \tag{3.10a}$$

$$f_k(t, x, v) = -\, m^{-1} \int_o^t dt\, F^{NSA}{}_k(t, x, v) \,/\, v_k, \tag{3.10b}$$

where there are no repeated indices, $\hat{m}$ is constant and the functions f_k are computed from Eq.s (3.10b). **q.e.d.**

The primary motivations for the submission of the isotopic Newton's equations are expressed by the following properties with self-evident proofs.

Corollary 1.A

The isotopic Newton equations permit a representation of the actual nonspherical shape of the body considered and of its possible deformations via the generalized unit (or isotopic element) of the theory.

Recall that Newton's [13] equations were based on the Galilean [5] approximate the body considered as a massive point. The point-like representation of particles then implied only action-at-a-distance, potential interactions with consequential analytic representations via Hamilton's equations as well as under symplectic map into quantum mechanical formulations. A representation of the extended character of particles is reached in *second* quantization via the form factors. However, this representation is restricted to spherical shapes from the fundamental symmetry of quantum mechanics, the rotational symmetry. Moreover, the latter symmetry is known to be a symmetry of *rigid* bodies. Form factors cannot therefore represent the *deformations* of particles under sufficiently intense external interactions which is studied via other rather complicated procedures.

A first motivation for the studies presented in this paper is to introduce a representation of actual *nonspherical and deformable* shapes of particles at the primitive *Newtonian* level, which then persists under *classical* analytic representations as well as under maps to *first* quantization. The isotopic Newton equations do indeed achieve these objectives by setting the foundations for possible new advances in classical and quantum physics. The objective is achieved via the new degrees of freedom of the generalized unit of the theory which are evidently absent in the conventional Newtonian, classical and quantum formulations.

As a simple case, suppose that the body considered is a rigid spheroidal ellipsoid with semiaxes n_1^2, n_2^2, n_3^2 = constants. Such a shape is directly represented by the isotopic element of the theory in the simple diagonal form (1.21), i.e.,

$$\hat{T} = \text{diag.}\,(\, n_1^{-2}, n_2^{-2}, n_3^{-2}\,),\quad n_k = \text{const} > 0,$$

$$k = 1, 2, 3,\quad \hat{T}_o^o = 1. \tag{3.11}$$

The representation of the shape in isospace $\hat{S}(\hat{t}, \hat{x}, \hat{v})$ is then embedded in the *isoderivatives* of the isotopic Newton equations and, when projected in the conventional space $S(t, x, v)$ can be written

$$m\, \hat{T}_k^i \frac{d\, v_i}{dt} - \hat{T}_k^i \frac{\partial U_i(t, x)}{\partial\, x^s} v^s + \hat{T}_k^i \frac{\partial U_o(t, x)}{\partial\, x^i} = 0, \tag{3.12}$$

namely, the shape terms $\hat{T}_k^i$ are admitted as factors.

Note that *the representation of shape occurs only in isospace because, when projected in the conventional Euclidean space, the shape terms cancel out by recovering the conventional point-like character of Newton's equations.* This illustrates the necessity of the isotopy for the representation of shape. Moreover, the *nonspherical character of the shape emerges only in the projection in ordinary spaces, because all deformed spheres in ordinary spaces are mapped into the perfect isosphere in isospace* (Sect. 1),

$$\hat{x}^2 = (\, x^1 n_1^{-2} x^1 + x^2 n_2^{-2} x^2 + x^3 n_3^{-2} x^3\,)\, \hat{1} \in \hat{R}(\hat{n}, +, \hat{\times})\,. \tag{3.13}$$

The representation of shapes more complex than the spheroidal ellipsoids is possible with non-diagonal isounits. The representation of the *deformations* of the original shape due to motion within resistive media or other reasons, can be achieved via a suitable functional dependence of the $\hat{T}_k^i$ terms in velocities, pressure, etc.

(see [19,20] for various applications in classical and quantum mechanics).

Corollary 1.B

The isotopic Newton equations permit a novel representation of variationally non-self-adjoint forces via the isometric of the underlying geometry, according to the rules

$$m\, dv_k / dt - F^{NSA}_k(t, x, v) = \hat{1}^i_k\, m\, d\hat{T}_i^j v_j / dt, \quad (3.14)$$

while leaving unchanged the representation of conventional self-adjoint forces (except for the constant factor $\hat{T}^o_o$ of U_k).

In fact, the non-self-adjoint forces are embedded in the *covariant* coordinates in isospace $\hat{v}_i = T_i^j v_j$, where the v_j are the covariant coordinates in conventional space. The novelty therefore lies on the fact that non-self-adjoint forces are represented by the isogeometry itself, thus providing another motivation for the isotopies.

The simplicity of representation (3.14) should be kept in mind and compared to the complexity of the conventional solution of the *inverse problem of Newtonian mechanics* [19,20], i.e., the representation of non-self-adjoint systems via a Lagrangian or a Hamiltonian. Moreover, under the assumed conditions, the latter exists in the fixed coordinates (t, x, v) of the observer only for a restricted class called *nonessentially nonselfadjoint* [loc. cit.], while isorepresentation (3.6) always exist in the given coordinates (t, x, v) under the same conditions. Physical problematic aspects in the use of coordinate transformations are discussed in the second paper of this series.

The following examples illustrate isorepresentation (3.6). The equation of the linearly damped particle in one dimension

$$m\, dv/dt + \gamma v = 0, \quad \gamma \in R(n,+,\times), \ \gamma > 0, \quad (3.15)$$

admits isorepresentation (3.6) with values

$$\hat{T} = \hat{S}_o e^{\gamma t/2m}, \quad \hat{T}_o^o = 1, \quad U_k = U_o = 0, \quad (3.16)$$

where $\hat{S}_o$ is a *shape factor*, e.g., of the spheroidal type (3.11) which is prolate in the direction of motion. In this way, the isotopic Newton equations represent: 1) the nonselfadjoint force $F^{NSA} = -\gamma v$ experienced by an object moving within a physical medium; 2) its extended character (which is necessary for the existence of the resistive force); and 3) the deformation of the original shape (in the case considered a perfect sphere) caused by the medium.

The equation for the linearly damped harmonic oscillator in one dimension

$$m\ddot{x} + \gamma\dot{x} + k x = 0, \quad k \in R(n,+,\times), \ k > 0, \quad (3.17)$$

admits isorepresentation (3.6) with the values

$$\hat{T} = \hat{S}_o e^{\gamma t/2m}, \quad U_o = -\tfrac{1}{2} k x^2, \ U_k = 0, \quad \hat{T}_o^o = 1, \quad (3.18)$$

where $\hat{S}_o$ represents the shape of the body oscillating within a resistive medium. The interested reader can construct a virtually endless variety of isorepresentations of non-self-adjoint forces. A systematic study will be conducted elsewhere.

Corollary 1.C

The isotopic Newton equations permit the representation of nonlocal-integral forces when completely embedded in the isounit of the theory.

The above occurrence is permitted by the integro-differential topology of isomanifolds $\hat{M}(\hat{E})$ recalled at the end of Sect. 1. Consider as an example the integro-differential equation

$$m\, dv/dt + \gamma v^2 \int_\sigma d\sigma\, \mathcal{F}(\sigma, \ldots) = 0, \quad \gamma > 0, \quad (3.19)$$

representing an extended object (such as a space-ship during re-entry in our atmosphere) with local-differential center-of-mass trajectory x(t) and corrective terms of integral type due to the shape (surface) σ of the body moving within a resistive medium, where $\mathcal{F}$ is a suitable kernel depending on σ as well as on other variables such as pressure, temperature, density, etc. The above equation admits isorepresentation (3.6) with the values

$$\hat{T} = \hat{S}_\sigma e^{\gamma m^{-1} x \int_\sigma d\sigma\, \mathcal{F}(\sigma, \ldots)}, \quad \hat{T}_o^o = 1, \quad U_k = U_o = 0, \quad (3.20)$$

where $\hat{S}_\sigma$ is the shape factor, which is admitted by the integro-differential topology of the isomanifold $\hat{M}(\hat{E})$ because all integral terms are embedded in the isounit.

Similar isorepresentations can be easily constructed by the interested reader.

It should be recalled that the representation of nonlocal-integral terms is prohibited in Hamiltonian mechanics because the underlying geometry and topology are local-differential. In fact, the Lie-Koening Theorem requires a *local-differential approximation* of systems and it is inapplicable to integral systems of type (3.19). Further developments and implications will be discussed in the subsequent papers.

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