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Normal index and Frattini type subgroups of finite groups

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Abstract

The normal index $\eta(G:M)$ of a maximal subgroup M of a finite group G is the order of a chief factor H/K where H is minimal in the set of normal supplements of M in G. This notion is used to identify the largest normal solvable subgroup of G as the intersection of members of certain class of maximal subgroups of G.

Key words: Solvable, supersolvable, p-solvable, p-supersolvable.

Indice normal y subgrupos tipo Frattini de grupos finitos

Resumen

El indice normal $\eta(G:M)$ de un subgrupo maximal M de un grupo finito G es el orden de un factor principal H/K donde H es minimal en el conjunto de normal suplimentos de M en G. Esta noción es usada para identificar el más grande subgrupo normal soluble de G como la intersección de miembros de ciertas clases de subgrupos maximal de G.

Palabras claves: Soluble, supersoluble, p-soluble, p-supersoluble.

Introduction

The class of maximal subgroups of a finite group together with the subclass of maximal subgroups with composite indices and the one in which the index of each member is composite but not divisible by a fixed prime p have been studied in detail in recent years [1-4]. Finite group structure in very much related to the normal indices of the members of these classes and the investigations of these required the consideration of the intersection of the members belonging to each of these classes. The study of the Frattini like subgroups thus obtained led to the sharpening of Huppert's Theorem which state that a finite group is supersolvable if and only if each maximal subgroup is of prime index [5].

In this note the class of maximal subgroups each of whose index is divisible by a fixed prime p and also the subclass consisting of members with composite indices are considered. These classes yield Frattini like subgroups in a natural way. Using the notion of normal index the study of other related Frattini like subgroups is further continued and the largest normal solvable subgroup of a finite group is identified as one Frattini like subgroup. This is the main result in the paper. We recall the definition of normal index from [6].

Definition [4.2.1]

The normal index of a maximal subgroup M of a finite group G is defined to be the order of a chief factor H/K where H is minimal in the set of normal supplements of M in G.

Normal index is uniquely defined and a detailed proof is given in **Lemma 1** in [7]. It also immediately follows from the definition that the index of a maximal subgroup always divides its normal index.

The following standard notations and terminologies have been used throughout.

a. M is a maximal subgroup G: $M<.G$

b. Normal index of maximal subgroup M: $\eta(G:M)$

c. The p-part of normal index: $\eta(G:M)_p$

d. The p-part of index of maximal subgroup M of G: $[G:M]_p$

e. $\phi_p(G)=\cap\{M<.G \mid [G:M]_p=1\}$

f. $\phi(G)=\cap\{M<.G\}$

g. A maximal subgroup whose index is a composite number: c-maximal subgroup.

Preliminary results

In this section we shall discuss a few basic results on the influences of indices and normal indices of maximal subgroups belonging to certain classes on finite group structures. The following lemmas are given for the sake of completeness. They have been used extensively throughout.

Lemma 1: [8 Lemma 3]

If G is a group with a maximal core free subgroup then the following are equivalent:

(i) There exists a unique minimal normal subgroup of G and there exists a common prime divisor of the indices of all maximal core free subgroup of G.

(ii) There exists a nontrivial solvable normal subgroup of G.

(iii) The indices of all maximal core free subgroups of G are powers of a unique prime.

Lemma 2: [7, Lemma 2]

If $N \trianglelefteq G$ and $N \subseteq M$ then $\eta(G/N:M/N) = \eta(G:M)$.

The following generalization of the Frattini argument to π-solvable groups will be used. Its proof is a direct consequence of P. Hall's extended Sylow theorems [9].

Lemma 3

Let G be a π-solvable group and $N \trianglelefteq G$. If H is a Hall π-subgroup of N, then $G = N_G(H)N$.

We begin our discussion by considering a simple characterization of a p-solvable group that depends on the normal indices of maximal subgroups belonging to a certain class.

Proposition 1

A group G is p-solvable if and only if $\eta(G:M)_p \neq 1$ implies $\eta(G:M) = [G:M]$ for $M<.G$.

Proof

Let G be p-solvable and N be a minimal normal subgroup of G. Suppose $M<.G$ and $\eta(G:M)p \neq 1$. If $M \supset N$ then by induction it follows that $\eta(G:M) = [G:M]$. If $N \not\subseteq M$ then $G = MN$ and $\eta(G:M)_p \neq 1$ implies N is an elementary abelian p-group. It now follows that $\eta(G:M) = [G:M]$.

Now suppose, $\eta(G:M)_p \neq 1$ implies $\eta(G:M) = [G:M]$, $M<.G$. If G is simple, $\eta(G:M)_p = |G|_p \neq 1$ and $|G| = \eta(G:M) = [G:M]$ implies $M = <e>$. G is therefore of prime order and is therefore p-solvable. Let N be a minimal normal subgroup of G and consider G/N. By induction G/N is p-solvable and if $K \neq N$ is another minimal normal subgroup of G then G/K is also p-solvable and therefore, $G \cong G/K\cap N$ is p-solvable. N may, therefore, be assume as the unique minimal normal subgroup of G. If p does not divide $|N|$ then, evidently G is p-solvable and one, therefore, takes $|N|$ is divisible by p. Also, if $N \subset \phi(G)$, then p-solvability of G/N will imply $G/\phi(G)$ is p-solvable and this means G is p-solvable. However, if $N \not\subset \phi(G)$ then $G = MN$, $M<G$ and $\eta(G:M)p = |N|_p \neq 1$. Hence $|N| = \eta(G:M) = [G:M]$. It now follows from **Lemma 1** that N is solvable. Consequently, G is p-solvable.

Corollary

G is solvable iff $\eta(G:M) = [G:M]$ $\forall$ $M<.G$.

Proof

If G is solvable then the result follows from proposition 1. The converse follows immediately by induction and **Lemma 1.**

Theorem 1

Let G be p-solvable, $T_p = \{M<.G \mid [G:M]_p \neq 1\}$ and $H = \cap\{M<.G \mid M \in T_p\}$. Then H is p-nilpotent.

Proof

Let N be a minimal normal subgroup of G in H and consider G/N. By induction H/N is p-nilpotent and let R/N be the normal p-complement in H/N. Then R is the required normal p-complement in H if N is a p'-group. One may therefore assume $|N| = p^{\alpha}$ and R = YN, where Y is a p-complement of N in R. Evidently Y is a p-complement in H as well.

If X<.G then either $[G:X]_p \neq 1$ or $[G:X]_p = 1$ and in either case $N \subset X$. Thus $N \subset \phi(G)$. Since $R \trianglelefteq G$, it follows by **Lemma 3** and the fact that R is p-solvable, $G = N_G(Y).N = N_G(Y).\phi(G) = N_G(Y)$.

Hence $Y \trianglelefteq G$ and is the required normal p-complement in H.

Corollary

$\cap\{M<.G \mid \eta(G:M)_p \neq 1\} = W_p$ is p-nilpotent.

The intersection S(G) of maximal subgroups with composite indices each of which is also prime to a fixed prime p has been investigated in detail in [1-4]. Here we will consider the case when each index is composite and is also divisible by p. The following proposition will be helpful in this investigation. Recall that a group is called p-supersolvable when each of its chief factor has either order p or has p'-order.

Proposition 2

Let H be a normal solvable subgroup of G. If $\phi(G) \subset H$ then H is p-supersolvable if and only if $H/\phi(G)$ is p-supersolvable.

Proof

One needs only to show that p-supersolvability of $H/\phi(G)$ implies H is p-supersolvable. Let N be a minimal normal subgroup of G contained in $\phi(G)$. By induction H/N is p-supersolvable and note that N is elementary abelian. Since the formation F of p-supersolvable groups is saturated [9], it follows by theorem 1.7, pp. 152-153 [10], that H = NT, T is an F-projector. If $g \in G$ then $N = H^g = NT^g$ and since the F-projectors are conjugate, it now follows that $G = N.N_G(T) = \phi(G).N_G(T)$. Consequently, $G = N_G(T)$ i.e. $T \trianglelefteq G$ and $H = N \times T$. Hence H is p-supersolvable.

Corollary

Let G be solvable. Then $G/\phi(G)$ is p-supersolvable implies G is p-supersolvable.

Let p be a prime and C_p be the class $\{M<.G \mid [G:M]_p \neq 1$ and $[G:M]$ is composite$\}$. The Frattini-like subgroup $L_p = \cap\{M<.G \mid M \in C_p\}$ will now be considered.

Theorem 2

Let G be solvable. Then $L_p = \cap\{M<.G \mid [G:M]_p \neq 1$, $[G:M]$ is composite$\}$ is p-supersolvable.

Proof

Let N be a minimal normal subgroup of G contained in L_p. By induction L_p/N is p-supersolvable. If $N \subset \phi(G)$ then by proposition 2, L_p is p-supersolvable. It may therefore be assumed $N \not\subset X<.G$ so that G = XN. Without any loss of generality N may be assumed to be a p-group (and therefore is elementary abelian) for if N is a p'-group then clearly L_p is p-supersolvable. Now [G:X] cannot be composite as otherwise $N \subset X$. Therefore $[G:X] = |N| = p$, a prime. This however implies that L_p is p-supersolvable.

Corollary

Let G be solvable. Then $L = \bigcap_{p \mid |G|} L_p$ is supersolvable.

Remarks

1. Let $S_p = \cap\{M<.G \mid [G:M]_p = 1$, $[G:M]$ is composite$\}$. Then $S_p \cap L_p = L(G)$ is supersolvable [11].
2. If the set $C_p = M<.G \mid \{[G:M]_p \neq 1$, $[G:M]$ is composite$\} = \phi$ then $L_p = G$ which implies that if every maximal subgroup of G is either of index p or p' then G is p-supersolvable.

If a group G is supersolvable then from corollary to proposition 1 it follows that $\eta(G:M) = [G:M]$ = a prime and therefore is square free. Set $S = \{M<.G \mid \eta(G:M)$ is not square free$\}$.

Theorem 3

$W = \cap\{M<.G \mid M \in S\}$ is supersolvable.

Proof

Observe that W is characteristic in G and let N be a minimal normal subgroup of G contained in W. By induction $\frac{W}{N}$ is supersolvable. If $N \subset \phi(G)$ then it follows that $\frac{W\phi(G)}{\phi(G)}$ is supersolvable and this implies that $W\phi(G)$ is supersolvable [6, theorem 9]. Consequently, W is supersolvable. On the other hand, if $N \not\subset \phi(G)$ then for some $M <\cdot G$, $N \not\subset M$ and $G = MN$. Evidently, $\eta(G:M) = |N|$ is square free and this implies N is of prime order. Therefore W is supersolvable.

Corollary

1. If $S = \phi$ i.e. $\eta(G:M)$ is square free $\forall M <\cdot G$ then G is supersolvable.
2. Let U be a normal supersolvable subgroup of G. Then W.U. is supersolvable.

Proof

If $W \cap U = e$ then clearly the assertion follows. Assume therefore, $W \cap U \neq e$ and let N be a minimal normal subgroup of G contained in $W \cap U$. If $N \not\subset \phi(G)$ then $G = MN$, $M <\cdot G$ and $\eta(G:M)$ is square free. This implies N is of prime order and by induction $\frac{WU}{N}$ is supersolvable. Hence WU is supersolvable. On the other hand, if $N \subseteq \phi(G)$ then $\frac{U.W\phi(G)}{\phi(G)}$ is supersolvable. It now follows, again from theorem 9 in [6], that $UW\phi(G)$ is supersolvable i.e. UW is supersolvable.

3. $W \subseteq \cap\{X \mid X$ is a maximal normal supersolvable subgroup of G$\}$.

We have seen that a group G is solvable implies $\eta(G:M) = [G:M]$ for each $M <\cdot G$ and conversely solvability of G follows if $\eta(G:M) = [G:M]$ for each $M <\cdot G$. It turns out that the equality of these indices is essential for core free maximal subgroups to guarantee solvability of G. It does not happen in general that for a core free maximal subgroup $\eta(G:M)$ and $[G:M]$ are equal. This is precisely why the condition of p-solvability was required in Proposition 1.

Largest normal solvable subgroup

The observation made in above leads to the consideration of the classes of maximal subgroups in each of which there is a mismatch of the normal index and the index of each one of its members and this yields a simple description of the largest normal solvable subgroup of a group G.

Let $C_1 = \{M <\cdot G \mid [G:M]_p = 1$, $[G:M]$ is composite and $\eta(G:M) \neq [G:M]$, p is the largest prime divisor of $|G|\}$.

$C_2 = \{M <\cdot G \mid [G:M]$ is composite and $\eta(G:M) \neq [G:M]\}$.

$C_3 = \{M <\cdot G \mid \eta(G:M) \neq [G:M]\}$.

and W_i, $i = 1,2,3$ respectively be the intersection of the members of C_1, C_2 and C_3.

Then W_i, $i = 1,2,3$ is a characteristic subgroup of G and $W_1 \supseteq W_2 \supseteq W_3$ and if any of the classes C_i, $i = 1,2,3$ is empty then one puts $G = W_i$.

The theorem proved below is used to identify the largest normal solvable subgroup.

Theorem 4

The subgroup $T = \cap\{M <\cdot G \mid M \in C_2\}$ is solvable

Proof

Let N be a minimal normal subgroup of G in T and by induction it follows that T/N is solvable and N is the only minimal subgroup of G contained in T. If K is another minimal normal subgroup of G, $\frac{X}{K} <\cdot \frac{G}{K}$, $\eta\left(\frac{G}{K}:\frac{X}{K}\right) \neq \left[\frac{G}{K}:\frac{X}{K}\right]$ and $\left[\frac{G}{K}:\frac{X}{K}\right]$ is composite then $X \in C_2$. Set $\frac{W}{K} = \cap\left\{\frac{X}{K} <\cdot \frac{G}{K} \,\middle|\, \left[\frac{G}{K}:\frac{X}{K}\right]\right.$ is composite and $\left.\eta\left(\frac{G}{K}:\frac{X}{K}\right) \neq \left[\frac{G}{K}:\frac{X}{K}\right]\right\}$. Then $W \supseteq T$ and $\frac{W}{K}$ is solvable by induction and contains $\frac{TK}{K}$. Consequently, $\frac{TK}{K} \cong T$ is solvable. One may therefore assume N to be the unique minimal normal subgroup of G. Let p be the largest prime divisor of $|G|$. If $N \subset \phi_p(G)$ then T is solvable since $\phi_p(G)$ is solvable [6]. Now suppose $N \not\subset \phi_p(G)$. Then $G = YN$, $Y <\cdot G$, $[G:Y]_p = 1$.

Claim $\eta(G:Y) = [G:Y]$. Suppose it is not true. Then $[G:Y]$ is not composite as otherwise $N \subseteq Y$. Thus $[G:Y] = q$, a prime, $q \neq p$. By representing G on the cosets of Y it follows that core $Y \neq 1$ as otherwise $|G|$ will have to divide q! which is impossible. Hence, $\eta(G:Y) \neq [G:Y]$ is false and $\eta(G:Y) = [G:Y] = |N|$ = a p'-number. If W is any other core free maximal subgroup of G then $G = WN$ and $[G:W]_p = 1$. By similar arguments as before, one concludes $\eta(G:W) = [G:W] = |N|$. It now follows from **Lemma 1** than N is solvable. Consequently, $\overline{T}$ is solvable.

Corollary

1. $W_3 = \cap\{M <. G \mid \eta(G:M) \neq [G:M]\}$ is solvable. If $C_3 = \phi$, $W_3 = G$ and $\eta(G:M) = [G:M]$ $\forall$ maximal subgroup M of G implies G is solvable.
2. If $C_2 = \phi$ then $T = G$ and $\eta(G:M) = [G:M]$ for each c-maximal subgroup M of G implies G is solvable.

Theorem 5

W_1 is solvable

Proof

Consider the class $J_p = \{M <. G \mid [G:M]_p = 1$ and $[G:M]$ is composite and, p is the largest prime divisor of $|G|\}$ and let $Sp(G) = \cap\{M <. G \mid M \in J_p\}$. By theorem 8 in [1] $S_p(G)$ is solvable. If $J_p = \phi$ then $G = S_p(G)$ is solvable and therefore W_1 is solvable. One may therefore assume $J_p \neq \phi$. If the class $C_1 = \phi$ then it implies that for each member M in J_p, $\eta(G:M) = [G:M]$. Then by theorem 3.1 in [4] G is solvable and consequently $W_1 = G$ is solvable. Thus one may assume $C_1 \neq \phi$.

Let N be a minimal normal subgroup of G in W_1 and consider G/N. By induction $\frac{W_1}{N}$ is solvable. If K is another minimal normal subgroup of G contained in W_1 then $\frac{W_1}{K}$ is solvable and therefore $\frac{W_1}{K \cap N} \cong W_1$ is solvable.

One may therefore treat N as the only minimal normal subgroup of G contained in W_1. Suppose $V \neq N$ is any other minimal normal subgroup of G. Consider $G/V = \overline{G}$ and let $X/V = \overline{X} <. \overline{G}$.

Set $A = \{\overline{X} <. \overline{G} \mid [\overline{G}:\overline{X}.]_p = 1$, X is c-maximal and $\eta(\overline{G}:\overline{X}) \neq [\overline{G}:\overline{X}]\}$

and

$B = \{\overline{X} <. G \mid \overline{X}$ is c-maximal and $\eta(\overline{G}:\overline{X}) \neq [\overline{G}:\overline{X}]\}$.

If p divides $|\overline{G}|$ then for $\frac{X}{V} = \overline{X} \in A$ it follows that $X \in C_1$ and if $\frac{T}{V} = \overline{T} = \cap\{\overline{X} <. \overline{G} \mid \overline{X} \in A\}$ then by induction it follows that T/V is solvable and $T \supseteq W_1$. Hence $\frac{W_1 V}{V}$ is solvable and therefore $\frac{W_1 V}{V} \cong \frac{W_1}{W_1 \cap V} \cong W_1$ is solvable.

On the other hand if p does not divide $|\overline{G}|$ then for any $X/V = \overline{X} \in B$, it follows that $X \in C_1$.

Let $\frac{R}{V} = \overline{R} = \cap\{\overline{X} <. \overline{G} \mid \overline{X} \in B\}$. By theorem 5, $\overline{R}$ is solvable and note $R \supseteq W_1$. As before it follows that W_1 is solvable and N may be treated as the unique minimal normal subgroup of G. Two cases need to be distinguished. Case I: $p \mid |N|$. Case II: $p \nmid |N|$.

Case I: $p \mid |N|$. One may assume that $N \not\subset \phi(G)$ and therefore there exists core free maximal subgroup X in G so that $G = XN$. If Y is any core free maximal subgroup then $G = YN$ and $[G:Y]_p = 1$ or $[G:Y]_p \neq 1$. In case $[G:Y]_p = 1$ then $[G:Y]$ is a composite number. For if $[G:Y] = q$, a prime then by representing G on the cosets of Y if follows that $|G|$ divides q!, a contradiction since q is not the largest prime divisor of $|G|$. Therefore, $[G:Y]$ is composite and it follows that $\eta(G:Y) = [G:Y]$ as otherwise N will be included in Y. Thus $|N| = \eta(G:Y) = [G:Y]$, and $p \mid [G:Y]$, a contradiction. One may therefore conclude p divides the index of every core free maximal subgroup of G and by lemma 1, N is solvable and therefore W_1 is solvable.

Case II. $p \nmid |N|$. Again one may assume $N \not\subset \phi(G)$ as otherwise the result follows immediately. If R is any maximal subgroup G, $N \subset R$ then $G = NR$ and $[G:R]_p = 1$. As in case I, $[G:R]$ cannot be a prime and $[G:R]$ is composite implies that $\eta(G:R) = [G:R]$. Thus $|N| = \eta(G:R) = [G:R]$ and there exists a common prime divisor of the indices of all core free maximal subgroups. Therefore N is solvable which implies W_1 is solvable.

Theorem 6

In any group G, W_3 is the largest normal solvable subgroup.

Proof

From theorem 5 it follows that W_3 is solvable. Let K be the largest normal solvable subgroup of G and V be a minimal normal subgroup of G contained in K. Evidently, V is elementary abelian. If $M \in C_3$ and $V \not\subset M$ then $G = MV$ and it follows that $\eta(G:M) = [G:M]$, a contradiction.

Hence $V \subset M \ \forall \ M \in C_3$ and therefore $\forall \subset W_3$. Consider $G/V = \overline{G}$ and if $\overline{X} = \frac{X}{V} <\cdot \frac{G}{V} = \overline{G}$ such that $\eta(G/V:X/V) \neq [G/V:X/V]$ then $\eta(G:X) \neq [G:X]$ and therefore $X \in C_3$. Since K/V is the largest normal solvable subgroup of G/V it follows by induction that $\frac{K}{V} = \cap\{\overline{X} <\cdot \overline{G} \mid \eta(\overline{G}:\overline{X}) \neq [\overline{G}:\overline{X}]\} = \frac{W_3}{V}$.

Therefore $K = W_3$ and the assertion is proved.

Corollary

$W_1 = W_2 = W_3$.

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