Regularity for an elliptic problem

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Abstract

In the present paper, regularity results for linear elliptic equations are presented, where a coefficient has critical growth.

Key words: Regularity, elliptic equation.

Regularidad para un problema elíptico

Resumen

En este trabajo se presenta un resultado de regularidad sobre una ecuación lineal elíptica donde un coeficiente tiene un crecimiento crítico.

Palabras claves: Regularidad, ecuación elíptica.

Introduction

The main purpose of this paper is to give a regularity result for a linear elliptic equation where a coefficient has critical growth. We consider the boundary value problem

$$\begin{aligned} -\Delta u(x) - qu &= \psi \quad in\ \Omega \\ u(x) &= 0 \quad on\ \partial\Omega \end{aligned} \qquad (1)$$

Where Δ is the Laplacian operator, Ω is some open subset of R^n such that the Sobolev imbedding theorem applies, $q \in L^\infty(\Omega) + L^p(\Omega)$, $\psi \in L^2(\Omega) \cap L^\infty(\Omega)$, and $\|q\|_{L^\infty} > \lambda_1$ where λ_1 is the first eigenvalue of $-\Delta$.

The motivation for proving Theorem A below came from the study of the problem

$$\begin{aligned} -\Delta u(x) &= g(\lambda, u(x)) &&, x \in B,\ u \in C^2(B) \\ u &> 0 &&, \text{in } B \\ u &= 0 &&, \text{on } \partial B \end{aligned}$$

with $g(\lambda,u) = |u|^r u + \lambda$, where $r = \dfrac{4}{N-2}$ and $N \geq 3$.

The main result for problem (1) is

Theorem A

Assume $q \in L^\infty(\Omega) + L^p(\Omega)$, where

$$p = \begin{cases} \dfrac{N}{2} & , \text{if } N \geq 3 \\ 1 & , \text{if } N = 1 \end{cases}, \text{ and } p > 1, \text{ if } N = 2.$$

If $\psi \in L^2(\Omega) \cap L^\infty(\Omega)$ and $u \in H_0^1(\Omega)$ is the unique solution of (1), then $u \in \bigcap_{2 \leq p < \infty} L^p(\Omega)$.

Theorem A is an extension of a theorem of Brezis and Kato [2]. The proof is based on the Sobolev imbedding theorem (see Adams [1]) and uses some ideas taken from [2]. Specifically, if $u \in H_0^1(\Omega)$ is the unique solution of (1), then we shall prove the existence of a sequence $\{u_k\}$ which converges weakly to u in $H_0^1(\Omega)$. The Sobolev imbedding theorem is then used to show that $u_k \in \bigcap_{2\le p<\infty} L^p(\Omega)$, implying that $u \in \bigcap_{2\le p<\infty} L^p(\Omega)$.

Proof of Theorem A

Lemma 2.1

Let $u \in H_0^1(\Omega)$. If for $k = 1, 2,...$ we let $u_k = \min(u,k)$, then $\{u_k\}$ converges weakly in $H_0^1(\Omega)$ to u.

Proof. From the definition of u_k we see that $\{u_k\}$ and $\{\nabla u_k\}$ converges pointwise to u and ∇u respectively. On the other hand if $\varphi \in H_0^1(\Omega)$, and $\Lambda_k = \{ x \in \Omega : u(x) \le k\}$, then

$$\int_\Omega \nabla u_k \cdot \nabla\varphi = \int_{\Lambda k} \nabla u \cdot \nabla\varphi \le \left(\int_{\Lambda_k} |\nabla u|^2\right)^{\frac{1}{2}} \left(\int_{\Lambda_k} |\nabla\varphi|^2\right)^{\frac{1}{2}} \le$$

$$\|u\|_{H_0^1(\Omega)} \|\varphi\|_{H_0^1(\Omega)}.$$

Thus by the dominating convergence theorem, $\int_\Omega \nabla u_k \cdot \nabla\varphi \to \int_\Omega \nabla u \cdot \nabla\varphi$. This completes the proof.

Lemma 2.2.

Let $q \in L^\infty(\Omega) + L^p(\Omega)$ with

$$p = \begin{cases} \frac{N}{2} & , \text{ if } N \ge 3 \\ 1 & , \text{ if } N = 1 \end{cases}, \quad \text{and } p > 1, \text{ if } N = 2.$$

Then for every $\epsilon > 0$, there exists a constant λ_θ such that

$$\int_\Omega q|u|^2 \le \epsilon \|grad\, u\|_{L^2}^2 + \lambda_\epsilon \|u\|_{L^2}^2, \quad \forall\, u \in H_0^1(\Omega).$$

Proof. Write $q = q_1 + q_2$ with $q_1 \in L^\infty(\Omega)$ and $q_2 \in L^p(\Omega)$. Then for every $k > 0$ we have

$$\int_\Omega q|u|^2 \le \|q_1\|_{L^\infty} \|u\|_{L^2}^2 + \int_{[|q_2|>k]} q_2|u|^2 + k \int_{[|q_2|\le k]} |u|^2$$

$$\le (\|q_1\|_{L^\infty} + k)\, \|u\|_{L^2}^2 + \|q_2\|_{L^p(|q_2|>k)} \|u\|_{L^t}^2,$$

where $\frac{1}{p} + \frac{2}{t} = 1$.

In case $N \ge 3$ we find $t = 2^*$ where 2^* is the Sobolev exponent, that is $2^* = \frac{2N}{N-2}$.

By the Sobolev imbedding theorem we have

$$\|u\|_{L^t} \le C\, \|grad\, u\|_{L^2}, \quad \forall u \in H_0^1(\Omega).$$

When $N = 2$ we find $2 < t < \infty$ and it is known that

$$\|u\|_{L^t} \le C\, (\|grad\, u\|_{L^2} + \|u\|_{L^2}), \quad \forall u \in H_0^1(\Omega).$$

When $N = 1$ we find $t = \infty$ and it is known that

$$\|u\|_{L^\infty} \le C\, (\|grad\, u\|_{L^2} + \|u\|_{L^2}), \quad \forall u \in H_0^1(\Omega).$$

Therefore we reach the conclusion of lemma 2.1 in all the cases by choosing k large enough so that $C^2\, \|q_2\|_{L^p(|q_2|>k)} < \epsilon$.

Now we come to the proof of Theorem A. We have only to consider the case $N \ge 3$ (when $N \le 2$, $u \in H_0^1(\Omega)$ implies $u \in \bigcap_{2\le p<\infty} L^p(\Omega)$).

We truncate q by $q_k = \min(q,k)$ and define u_k to be the unique solution of

$$u_k \in H_0^1(\Omega)$$
$$-\Delta u_k - q_k u_k = \psi \text{ in } \Omega$$

We shall prove that for every $p \in [2,\infty)$, $u_k \in L^p(\Omega)$ and

$$\|u_k\|_{L^p} \leq C_p(\|\psi\|_{L^2} + \|\psi\|_{L^\infty}, \quad (2)$$

where C_p is independent of k, but it depends on q through the use of lemma 2.2 for simplicity we drop now the subscript k on u_k and write

$$-\Delta u - q_k u = \psi. \quad (3)$$

Set $u_m = \min(u,m)$ and let $2 \leq p < \infty$; since $(u_m)^{p-1} \in H_0^1(\Omega)$ we can multiply (3) by $(u_m)^{p-1}$ and obtain

$$(p-1)\int_\Omega |grad\, u_m|^2 (u_m)^{p-2} \leq \int_\Omega \psi(u_m)^{p-1} + \int_\Omega qk(u_m)^p.$$

That is

$$\frac{4(p-1)}{p^2}\int_\Omega |grad\,(u_m)^{\frac{p}{2}}|^2 \leq \|\psi\|_{L^p}\|u_m\|_{L^p}^{p-1} + \int_\Omega q(u_m)^p$$

$$\leq \|\psi\|_{L^p}\|u_m\|_{L^p}^{p-1} + \epsilon\|grad\,(u_m)^{\frac{p}{2}}\|_{L^2}^2 + \lambda_\epsilon\|u_m\|_{L^p}^p.$$

By choosing $\epsilon > 0$ small enough (for example $\epsilon = \frac{2(p-1)}{p^2}$), we see that

$$\int_\Omega |grad\,(u_m)^{\frac{p}{2}}|^2 \leq C_p(\|\psi\|_{L^p}^p + \|u\|_{L^p}^p),$$

were C_p is independent of k and m.

Using Sobolev's inequality we see that

$$\|u\|_{L^{\frac{2*p}{2}}}^p \leq C_p(\|\psi\|_{L^p}^p + \|u\|_{L^p}^p). \quad (4)$$

Thus if $u \in L^p(\Omega)$ then $u \in L^{\frac{2*p}{2}}(\Omega)$ and

$$\|u\|_{L^{\frac{2*p}{2}}}^p \leq C_p(\|\psi\|_{L^p} + \|u\|_{L^p}).$$

Iterating the process from $p = 2$ we obtain finally that for every $p \in [2,\infty)$

$$\|u\|_{L^p} \leq C_p(\|\psi\|_{L^2} + \|\psi\|_{L^\infty}).$$

More precisely we have proved (2).

References

1. Adams, R.: Sobolev Spaces. Academic Press, 1975.
2. Brezis, H. and Kató, T.: Remarks on the Schrodinger operator with singular complex potentials. J. Math. Pures et Appl., 58, 1979, pp. 137-151.

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