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# ON A q-ANALOGUE OF HERMITE POLYNOMIALS

## ABSTRACT

The present paper attempts at exploring some properties for a q-analogue of Hermite polynomials.

## RESUMEN

El presente trabajo trata de explorar algunas propiedades para una q-análoga de los polinomios de Hermite.

## INTRODUCTION

For $|q| < 1$, the q-shifted factorials are defined by

$$(a;q)_n = \begin{cases} 1, & \text{if } n = 0 \\ (1-a)(1-aq)\dots(1-aq^{n-1}), & \text{if } n=1,2,\dots, \end{cases}$$

with $(a;q)_\infty = \prod_{n=0}^{\infty} (1-aq^n)$.

For convenience, we shall write $[a]_n$ to mean $(a;q)_n$.

Let $\delta_x$ be the q-derivative defined by

$$\delta_x f(x) = \frac{1}{x} \{ f(x) - f(xq) \}. \tag{1.1}$$

For brevity, we shall write $\delta$ for $\delta_x$.

The generalized basic hypergeometric series (see [3, p.347])

$${}_{p+1}\phi_{p+r}\left[\begin{matrix} a_1,\dots,a_{p+1};q,z \\ b_1,\dots,b_{p+r} \end{matrix}\right] =$$

$$\sum_{n=0}^{\infty} (-1)^{rn}\, q^{rn(n-1)/2}\, \frac{[a_1]_n\dots[a_{p+1}]_n\, z^n}{[b_1]_n\dots[b_{p+r}]_n [q]_n}$$

In what follows, the other notations and definitions carry their usual meaning.

Consider a q-analogue of Hermite polynomials:

$$H_{n,q}(x) = \sum_{j=0}^{[n/2]} (1)^j \frac{[q^{-n}]_{2j}}{(q^2;q^2)_j} x^{n-2j}. \tag{1.2}$$

It is not difficult to see

$$\lim_{q\to 1^-} 2^n(1-q^2)^{-n/2} H_{n,q}(x\sqrt{1-q^2}\,) = H_n(x), \tag{1.3}$$

where $H_n(x)$ denotes the usual set of classical Hermite polynomials.

It is fairly easy to verify after reverting the order of summation in (1.2) for even and odd integers that

$$H_{2n,q}(x) =$$

$$(-1)^n (q^2;q^2)_n\, q^{-n(1+2n)}\, L_n^{(-1/2)}(x^2q^2|q^2); \tag{1.4}$$

$$H_{2n+1,q}(x) =$$

$$(-1)^n (q^2;q^2)_n\, q^{-n(3+2n)} x L_n^{(1/2)}(x^2q^2|q^2), \tag{1.5}$$

where $L_n^{(\alpha)}(x|q) = \frac{[q^{1+\alpha}]_n}{[q]_n} {}_1\phi_1 \left[ \begin{matrix} q^{-n}, q, xq^{1+\alpha+n} \\ q^{1+\alpha} \end{matrix} \right]$

(1.6)

are the q-Laguerre polynomials of Hahn [1] satisfying the orthogonality condition (see Moak [2])

$$\int_0^\infty L_m^{(\alpha)}(x|q)\, L_n^{(\alpha)}(x|q) \frac{x^\alpha}{[-x]_\infty} dx$$

$$= \frac{\Gamma(1+\alpha)\Gamma(-\alpha)[q^{1+\alpha}]_n}{\Gamma_q(-\alpha)[q]_n} q^{-n} (1-q)^{1+\alpha} \delta_n^m \quad (1.7)$$

where $\alpha > -1$ and $\delta_n^m$ is the familiar Kronecker delta.

## 2. ORTHOGONALITY

Using (1.7), in virtue of (1.4) and (1.5), one readily obtains the following orthogonality condition for the polynomials $H_{n,q}(x)$:

$$\int_{-\infty}^\infty H_{n,q}(x)\, H_{m,q}(x) \frac{dx}{(-x^2q^2;q^2)_\infty}$$

$$= \pi \sqrt{1-q^2}\, [q]_n\, q^{-(1+n)^2} \delta_n^m / \Gamma_{q^2}(1/2).$$

## 3. SOME PROPERTIES

From the explicit representation (1.2), we can obtain by usual series techniques; in view of q-binomial theorem [3, p.348, eq.(274)] and the elementary identity

$$[aq^{-n}]_\infty = (-1)^n a^n q^{-n(n+1)/2} [q/a]_n [a]_\infty ,$$

the following generating function:

$$\sum_{n=0}^\infty \frac{[c]_n}{[q]_n} H_{n,q}(x)t^n = \frac{[cxt]_\infty}{[xt]_\infty} {}_3\phi_2 \left[ \begin{matrix} 0, c, cq; q^2, -1/x^2 \\ q/xt,\ q^2/xt \end{matrix} \right]$$

(3.1)

In (3.1) replacing t by t/c and taking limit as $c \to \infty$ and changing again t by -tq, we get the generating function

$$\sum_{n=0}^\infty q^{n(n+1)/2} H_{n,q}(x) \frac{t^n}{[q]_n} = [-xtq]_\infty / [-t^2;q^2]_\infty \quad (3.2)$$

In (3.2) replacing x by xy and in the right hand side of resulting identity expanding the term $(-y^2t^2;q^2)_\infty/(-t^2;q^2)_\infty$ by q-binomial theorem and then equating the coefficients of $t^n$ on both sides, we get a multiplication formula

$$H_{n,q}(xy) = \sum_{j=0}^{[n/2]} (-1)^j \frac{[q^{-n}]_{2j}}{(q^2;q^2)_j} y^{n-2j} (y^2;q^2)_j H_{n-2j,q}(x).$$

Since $[x]_\infty = \sum_{n=0}^\infty (-1)^n q^{n(n-1)/2} \frac{x^n}{[q]_n}$,

from (3.2), by routine method, we have expansion of $x^n$ in the form

$$x^n = \sum_{r=0}^{[n/2]} \frac{[q]_n}{(q^2;q^2)_r [q]_{n-2r}} q^{r(3r-2n-2)} H_{n-2r,q}(x).$$

Again, from the generating function (3.2)

$$\delta \sum_{n=0}^\infty q^{n(n+1)/2} H_{n,q}(x) \frac{t^n}{[q]_n} = tq \frac{[-xtq^2]_\infty}{(-t^2;q^2)_\infty}, \quad (3.3)$$

so that

$$\sum_{n=0}^\infty q^{n(n+1)/2} \delta H_{n,q}(x) \frac{t^n}{[q]_n} =$$

$$tq \sum_{n=0}^\infty q^{n(n+1)/2} H_{n,q}(xq) \frac{t^n}{[q]_n},$$

which yields $\delta H_{0,q}(x) = 0$, and for $n \geq 1$,

$$\delta H_{n,q}(x) = (1-q^n)q^{1-n} H_{n-1,q}(xq) \quad (3.4)$$

or, more generally,

$$\delta^m H_{n,q}(x) = [q^{n-m-1}]_m q^{m(m-n)} H_{n-m,q}(xq^m), \quad n \geq m \geq 0.$$

Alternatively, we may write (3.3) as

$$\delta \sum_{n=0}^{\infty} q^{n(n+1)/2} H_{n,q}(x)(1+t^2) \frac{t^n}{[q]_n}$$

$$= tq \sum_{n=0}^{\infty} q^{n(n+1)/2} H_{n,q}(x) \frac{t^n q^n}{[q]_n},$$

which readily yields,

$$\delta H_{n,q}(x) + q^{1-2n} (1-q^{n-1}) (1-q^n) \delta H_{n-2,q}(x) =$$

$$(1-q^n)H_{n-1,q}(x).$$

It is easy to verify that

$$(x - \frac{t}{q})\delta \left\{ \frac{[-xtq]_\infty}{(-t^2;q^2)_\infty} \right\} - t\,\delta_t \left\{ \frac{[-xtq]_\infty}{(-t^2;q^2)_\infty} \right\} = 0,$$

which, in conjuntion with (3.2), gives

$$xq^{1+n} \delta H_{n,q}(x) = q^{1+n} (1-q^n)H_{n,q}(x) + (1-q^n) \delta H_{n-1,q}(x) \qquad (3.5)$$

Using (3.4) in (3.5), we have

$$q^n H_{n,q}(x) - xqH_{n-1,q}(xq) + q^{1-n}(1-q^{n-1})H_{n-2,q}(xq) = 0. \qquad (3.6)$$

From (3.4), in view of definition (1.1), we get

$$q^n H_{n,q}(x) = q^n H_{n,q}(xq) + xq(1-q^n) H_{n-1,q}(xq). \qquad (3.7)$$

Notice, however, that by eliminating the term $q^n H_{n,q}(x)$ between (3.6) and (3.7), and then changing x to x/q, we at once obtain the pure recurrence relation

$$H_{n,q}(x) - xH_{n-1,q}(x) + q^{1-2n}(1-q^{n-1})H_{n-2,q}(x) = 0.$$

Further, in (3.5) replacing x by xq and appealing to (3.4), we get a q-difference equation for q-Hermite polynomials in the form:

$$\delta^2 H_{n,q}(x) - xq^3 \delta H_{n,q}(xq) + q^3(1-q^n)H_{n,q}(xq) = 0. \qquad (3.8)$$

Following Moak [2,p.34], on account of (3.8), it is not hard to establish

$$\delta^2 u(x) + \left\{ \frac{1+q}{qx^2} - \frac{(1+q+x^2q^{4+n})(-x^2q^6;q^4)_\infty}{qx^2(-x^2q^4;q^4)_\infty} \right\} u(xq) = 0,$$

where $u(x) = H_{n,q}(x)/(-x^2q^4;q^4)_\infty$

Letting $b \to 0$ in the q-extension of Euler's transformation [3,p.348, eq.(281)], we obtain

$${}_2\phi_1 \left[ \begin{matrix} 0,a;q,z \\ c \end{matrix} \right] = \frac{1}{[z]_\infty} {}_1\phi_1 \left[ \begin{matrix} c/a;\ q,\ -az \\ c \end{matrix} \right], \qquad (3.9)$$

which is a q-extension of Kummer's first formula [3,p.322,eq.(183)].

Since $\frac{1}{[x]_\infty} = \sum_{n=0}^{\infty} \frac{x^n}{[q]_n}$, we have

$$\delta^{2n} \left\{ \frac{1}{(-x^2q^2;q^2)_\infty} \right\} = \sum_{r=n}^{\infty} \frac{(-1)^r q^{2r}}{(q^2;q^2)_r} x^{2r-2n} [q^{2r-2n+1}]_{2n},$$

which on replacing r by n+r and doing some straightforward manipulations reduce to

$$\delta^{2n} \left\{ \frac{1}{(-x^2q^2;q^2)_\infty} \right\} =$$

$$(-1)^n q^{2n}(q;q^2)_n \; {}_2\phi_1 \left[ \begin{matrix} 0,q^{1+2n};q^2,-x^2q^2 \\ q \end{matrix} \right] \qquad (3.10)$$

Now, by applying (3.9) to (3.10) and then using (1.4) and (1.6), we find that

$$H_{2n,q}(x) = (-1)^{2n} q^{-3n-2n^2}(-x^2q^2;q^2)_\infty \delta^{2n} \left\{ \frac{1}{(-x^2q^2;q^2)_\infty} \right\} \qquad (3.11)$$

Similarly, we can obtain

$$H_{2n+1,q}(x) =$$

$$(-1)^{2n+1} q^{-2-5n-2n^2} (-x^2q^2;q^2)_\infty \, \delta^{2n+1} \left\{ \frac{1}{(-x^2q^2;q^2)_\infty} \right\} \tag{3.12}$$

Combining (3.11) and (3.12), we get a Rodrígues type representation for the polynomials $H_{n,q}(x)$ as

$$H_{n,q}(x) = (-1)^n q^{-n(n+3)/2} (-x^2q^2;q^2)_\infty \, \delta^n \left\{ \frac{1}{(-x^2q^2;q^2)_\infty} \right\}.$$

More generally, one can obtain

$$\delta^k \left\{ \frac{H_{n,q}(x)}{(-x^2q^2;q^2)_\infty} \right\} =$$

$$(-1)^k q^{nk+(1/2)k(3+k)} \frac{H_{n+k,q}(x)}{(-x^2q^2;q^2)_\infty},$$

which, for k=1, reduces to a recurrence relation

$$xq^{n+2}H_{n+1,q}(x) + H_{n,q}(x) - (1+x^2q^2)H_{n,q}(xq) = 0.$$

One notes that all the properties for the polynomials $H_{n,q}(x)$ reduce, in view of (1.3), to corresponding properties for the Hermite polynomials $H_n(x)$.

## REFERENCES


[1] HAHN, W.: "Uber orthogonalpolynome, die q-differenzengleichungen genügen". Math.Nachr. 2(1949), 4-34.

[2] MOAK, D.S.: "The q-analogue of the Laguerre polynomials". J. Math. Anal. Appl. 81(1981), 20-47.

[3] SRIVASTAVA, H.M. and KARLSSON, P.W.: "Multiple Gaussian Hypergeometric Series", Halsted Press (Ellis Horwood Limited, Chichesler), John Wiley and Sons, New York (1985).


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