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CERTAIN FINITE EXPANSIONS ASSOCIATED WITH A BASIC ANALOGUE OF THE G-FUNCTION

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ABSTRACT

In this paper the authors derive certain finite expansion formulae associated with a basic analogue of Meijer's G-function. The results proved in this paper are the extensions of the results given earlier by R. P. Agarwal [1] and N. Agarwal [2].

RESUMEN

En este trabajo los autores derivan ciertos desarrollos finitos asociados con una análoga básica de la función G de Meijer. Los resultados probados en este trabajo son las extensiones de los resultados dados anteriormente por R.P. Agarwal [1] y N. Agarwal [2].

INTRODUCTION AND PRELIMINARIES

Recently, Saxena, Modi and Kalla [12] defined the basic analogue of the G-function in the following manner:

$$G_{A,B}^{m_1,n_1}\left[z;q\left|\begin{matrix}a_1,\dots,a_A\\ b_1,\dots,b_B\end{matrix}\right.\right]=$$

$$\frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{m_1} G(q^{b_j-\xi})\prod_{j=1}^{n_1} G(q^{1-a_j+\xi})\pi z^{\xi}}{\prod_{j=m_1+1}^{B} G(q^{1-b_j+\xi})\prod_{j=n_1+1}^{A} G(q^{a_j-\xi})G(q^{1-\xi})\sin\pi\xi}\,d\xi \tag{1}$$

where $0 \le m_1 \le B$, and $0 \le n_1 \le A$,

$$G(q^a) = \left\{\prod_{n=o}^{\infty}(1-q^{a+n})\right\}^{-1}; \tag{2}$$

The conventional notations (a) and (a)+m will be employed to represent the following sequences of A parameters as mentioned below:

$$(a) \equiv a_1,\dots,a_A;\ (a) + m \equiv a_1 + m,\dots,a_A + m$$

Here the contour 'C' is a line parallel to $R_l(\omega\xi) = 0$ with identations, if necessary, in such a manner, that all the poles of $G(q^{b_j-\xi})$ for $j=1,\dots,m_1$ are to the right and those of $G(q^{1-a_j+\xi})$ for $j=1,\dots,n_1$ to the left of it.

The integral converges, if

$$R_l\ [\xi\log(z) - \log\sin\pi\xi] < 0$$

for large values of $|\xi|$ on the contour, that is, if

$\left|\left\{\arg(z) - W_2W_1^{-1}\log|z|\right\}\right| < \pi$, where $|q| < 1, \log q = -W = -(W_1 + iW_2)$, W, W_1, W_2 are definite quantities, W_1 and W_2 being real.

Let $(a;q)_\mu = \prod_{j=o}^{\infty}\frac{(1-aq^j)}{(1-aq^{\mu+j})}$ for arbitrary μ and a,

so that

$$(a;q)_n = \begin{cases} 1, & \text{for } n = 0\\ (1-a)(1-aq)\dots(1-aq^{n-1}); & \forall\ n \in \{1,2,3,\dots\}\end{cases} \tag{3}$$

and

and

$$ {}_2\phi_1\left[a, q^{-n}; c; q, cq^{n}/a\right] = \frac{(c/a;\, q)_n}{(c;q)_n} \tag{17} $$

the well poised ${}_6\phi_5$, Slater [13]

$$ {}_6\phi_5\left[\begin{matrix} a,\ q\sqrt{a},\ -q\sqrt{a},\ b,\ c,\ q^{-N} \\ \sqrt{a},\ -\sqrt{a},\ aq/b,\ aq/c, aq^{1+N}; \end{matrix}\ q,\ aq^{1+N}/bc\right] = $$

$$ = \frac{(aq;\ q)_N\ (aq/bc;\ q)_N}{(aq/b;\ q)_N\ (aq/c;\ q)_N} \tag{18} $$

and the q-analogue of Dixon's theorem, namely,

$$ {}_4\phi_3\left[\begin{matrix} a,\ -q\sqrt{a},\ b,\ q^{-n} & ; \\ -\sqrt{a},\ aq/b,\ aq^{1+n}; & \end{matrix}\ q,\ \sqrt{a}\ q^{1+n}/b\right] $$

$$ = \frac{(aq;\ q)_n\ (\sqrt{a}\ q/b;\ q)_n}{(\sqrt{a}\ q;q)_n\ (aq/b;\ q)_n} \tag{19} $$

SPECIAL CASES

(i) If we set $B = m_1 = r$; $n_1 = 0$; $A = s$ and replace z by $\frac{-1}{x(1-q)^{s+1-r}}$ in (10) through (14), and apply the relation

$$ G^{r,0}_{s,r}\left[\frac{-1}{y}\ ;\ q \middle| \begin{matrix} (a) \\ (b) \end{matrix}\right] = \Pi\left[\begin{matrix} q,\ q^{a_1},\ldots,\ q^{a_s}\ ; \\ q^{b_1},\ldots,\ q^{b_r}\ \ ; \end{matrix}\right] $$

$$ \cdot\, {}_r\phi_s\left[\begin{matrix} b_1,\ldots,b_r; \\ a_1,\ldots,a_s; \end{matrix}\ q,y\right] \tag{20} $$

where $y = x(1-q)^{s+1-r}$, which follows from the integral given by Watson [14], we obtain the following results.

$$ \sum_{m=0}^{n} \frac{(q^{-n};q)_m\,(b_1-b_2-n;\ q)_m\ (b_3;q)_m \ldots (b_r;q)_m\ (yq^{b_2+n})^m}{(q;q)_m\ (a_1;q)_m \ldots (a_s;q)_m} $$

$$ \cdot\, {}_r\phi_s\left[\begin{matrix} b_1-n+m,\ b_2+n,\ b_3+m,\ldots,b_r+m; \\ a_1+m,\ \ldots,\ a_s+m\ \ ; \end{matrix}\ q,\ y\right] $$

$$ = {}_r\phi_s\left[\begin{matrix} b_1,\ldots,b_r; \\ a_1,\ldots,a_s; \end{matrix}\ q,y\right] \tag{21} $$

where $y = x(1-q)^{s+1-r}$

$$ \sum_{m=0}^{n} \frac{(q^{-n};q)_m\ (b_1;q)_m (b_3;q)_m \ldots (b_r;q)_m \left[yq^{b_2+n}\right]^m}{(q;\ q)_m\ (a_1;\ q)_m \ldots (a_s;\ q)_m} $$

$$ \cdot\, {}_r\phi_s\left[\begin{matrix} b_1+m,\ b_2+n,\ b_3+m,\ldots,b_r+m; \\ a_1+m,\ \ldots,\ a_s+m\ \ ; \end{matrix}\ q,\ y\right] $$

$$ = {}_r\phi_s\left[\begin{matrix} b_1,\ldots,b_r; \\ a_1,\ldots,a_s; \end{matrix}\ q,y\right] \tag{22} $$

where $y = x(1-q)^{s+1-r}$

$$\sum_{m=0}^{n} \frac{(q^{-n};q)_m (b_2;q)_m \dots (b_r;q)_m \, (yq^n)^m}{(q;q)_m (a_1;q)_m \dots (a_s;q)_m}$$

$$\cdot {}_r\phi_s \left[\begin{matrix} b_1+n, \; b_2+m, \dots, b_r+m; \\ a_1+m, \; \dots, a_s+m \; ; \end{matrix} \; q, \; y \right]$$

$$= {}_r\phi_s \left[\begin{matrix} b_1, \dots b_r; \\ a_1, \dots a_s; \end{matrix} \; q, y/q^n \right] \qquad (23)$$

where $y = x(1-q)^{s+1-r}$

$$\sum_{m=0}^{n} \frac{(q^{-n};q)_m (a;q)_m (1+\frac{a}{2};q)_m (-1-\frac{a}{2};q)_m (b;q)_m}{(q;q)_m (a/2;q)_m (1+a-b+n;q)_m (-a/2;q)_m}$$

$$\cdot \frac{(b_1;q)_m \dots (b_r;q)_m (-y)^m \, q^{\frac{m}{2}(m+1)+(a-b+n)m}}{(a_1;q)_m \dots (a_s;q)_m \, (1+a;q)_{2m}}$$

$$\cdot {}_{r+2}\phi_{s+2} \left[\begin{matrix} 1+a-b+m, \; 1+a+n+m, b_1+m, \dots, b_r+m \\ a_1+m, \dots, a_s+m, 1+a-b+n+m, 1+a+2m \; ; \end{matrix} \; q, \; y \right]$$

$$= {}_r\phi_s \left[\begin{matrix} b_1, \dots, b_r; \\ a_1, \dots, a_s; \end{matrix} \; q, \; y \right] \qquad (24)$$

where $y = x(1-q)^{s+1-r}$

$$\sum_{m=0}^{n} \frac{(q^{-n};q)_m (a;q)_m (-1-\frac{a}{2};q)_m (b_1;q)_m \dots (b_r;q)_m}{(q;q)_m (-a/2;q)_m (a_1;q)_m \dots (a_s;q)_m (1+a;q)_{2m}} (-y)^m q^{\frac{m}{2}(m+1)+(\frac{a}{2}+n)m}$$

$${}_{r+1}\phi_{s+1} \left[\begin{matrix} 1+a+n+m, \; b_1+m, \dots, b_r+m; \\ a_1+m, \dots, a_s+m, 1+a+2m \;\; ; \end{matrix} \; q, \; y \right]$$

$$= {}_{r+1}\phi_{s+1} \left[\begin{matrix} 1+\frac{a}{2}+n, \; b_1, \dots, b_r; \\ a_1, \dots, a_s, \; 1+\frac{a}{2} \;\; ; \end{matrix} \; q, \; y \right] \qquad (25)$$

where $y = x(1-q)^{s+1-r}$

(ii) On the other hand if we take $m_1 = B = r$; $n_1 = 0$ and $A = s$ in (10) and (13) and use the identity, Saxena, et al [12]

$$G^{r,o}_{s,r} \left[z; q \middle| \begin{matrix} (a) \\ (b) \end{matrix} \right] = E_q(r; b_j : s; a_t : z) \qquad (26)$$

We arrive at the results given earlier by N. Agarwal, [1].

(iii) Next, if we take $m_1 = B = 2$ and $n_1 = 0 = A$ in (10), it gives a result due to R.P. Agarwal, [2].
If we further take n=1; (10) then reduces to another result given by Agarwal [2].

(iv) Finally, if we take $n = 1$; $m_1 = B = r$; $n_1 = 0$ and $A = s$ in (10) and (11), the known recurrence relations proved by Agarwal [1] are obtained.

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REFERENCES

[1] AGARWAL, N.: "A q-analogue of MacRobert's generalized E-function". Ganita, 11, 1960, 49-63.

[2] AGARWAL, R.P.: "A basic analogue of MacRobert's E-function". Proc. Glasgow Math. Assoc., 5, 1961, 4-7.

[3] AGARWAL, R.P.: "Generalized hypergeometric series". Asia Publications House, London, 1963.

[4] ASKEY, R.: "The q-gamma and q-beta functions". Applicable Anal. 8(1978), 125-141.

[5] BAILEY, W.N.: "Generalized hypergeometric series". Cambridge University Press, 1935.

[6] ERDELYI, A. ET AL. : "Higher transcendental functions". Vol. I, McGraw Hill , New York, 1953.

[7] EXTON, H.: "Q-hypergeometric functions and applications". Halsted Press, Ellis Horwood, Chichester, John Wiley and Sons, New York, 1983.

[8] GASPER, G. and RAHMAN, M.: "Basic hypergeometric series". Cambridge University Press (to appear).

[9] LUKE, Y.L.: "The special functions and their approximations". Academic Press, New York, 1969.

where $L_n^{(\alpha)}(x|q) = \frac{[q^{1+a}]_n}{[q]_n} \; {}_1\phi_1 \left[\begin{matrix} q^{-n}, \; q, xq^{1+\alpha+n} \\ q^{1+\alpha} \end{matrix} \right]$

(1.6)

are the q-Laguerre polynomials of Hahn [1] satisfying the orthogonality condition (see Moak [2])

$$\int_0^\infty L_m^{(\alpha)}(x|q)\, L_n^{(\alpha)}(x|q)\, \frac{x^\alpha}{[-x]_\infty}\, dx$$

$$= \frac{\Gamma(1+\alpha)\Gamma(-\alpha)[q^{1+\alpha}]_n}{\Gamma_q(-\alpha)[q]_n}\, q^{-n}\,(1-q)^{1+\alpha}\, \delta_n^m \tag{1.7}$$

where $\alpha > -1$ and δ_n^m is the familiar Kronecker delta.

2. ORTHOGONALITY

Using (1.7), in virtue of (1.4) and (1.5), one readily obtains the following orthogonality condition for the polynomials $H_{n,q}(x)$:

$$\int_{-\infty}^{\infty} H_{n,q}(x)\cdot H_{m,q}(x)\, \frac{dx}{(-x^2q^2;q^2)_\infty}$$

$$= \pi\sqrt{1-q^2}\, [q]_n\, q^{-(1+n)^2}\, \delta_n^m / \Gamma_{q2}(1/2).$$

3. SOME PROPERTIES

From the explicit representation (1.2), we can obtain by usual series techniques; in view of q-binomial theorem [3, p.348, eq.(274)] and the elementary identity

$$[aq^{-n}]_\infty = (-1)^n a^n q^{-n(n+1)/2} [q/a]_n [a]_\infty ,$$

the following generating function:

$$\sum_{n=0}^{\infty} \frac{[c]_n}{[q]_n} H_{n,q}(x)t^n = \frac{[cxt]_\infty}{[xt]_\infty}\; {}_3\phi_2 \left[\begin{matrix} 0, c, cq; q^2, -1/x^2 \\ q/xt,\; q^2/xt \end{matrix} \right] \tag{3.1}$$

In (3.1) replacing t by t/c and taking limit as $c\to\infty$ and changing again t by -tq, we get the generating function

$$\sum_{n=0}^{\infty} q^{n(n+1)/2} H_{n,q}(x) \frac{t^n}{[q]_n} = [-xtq]_\infty \,/\, [-t^2;q^2]_\infty \tag{3.2}$$

In (3.2) replacing x by xy and in the right hand side of resulting identity expanding the term $(-y^2t^2;q^2)_\infty/(-t^2;q^2)_\infty$ by q-binomial theorem and then equating the coefficients of t^n on both sides, we get a multiplication formula

$$H_{n,q}(xy) = \sum_{j=0}^{[n/2]} (-1)^j \frac{[q^{-n}]_{2j}}{(q^2;q^2)_j}\, y^{n-2j}\, (y^2;q^2)_j\, H_{n-2j,q}(x).$$

Since $[x]_\infty = \sum_{n=0}^{\infty} (-1)^n q^{n(n-1)/2} \frac{x^n}{[q]_n}$,

from (3.2), by routine method, we have expansion of x^n in the form

$$x^n = \sum_{r=0}^{[n/2]} \frac{[q]_n}{(q^2;q^2)_r [q]_{n-2r}}\, q^{r(3r-2n-2)}\, H_{n-2r,q}(x).$$

Again, from the generating function (3.2)

$$\delta \sum_{n=0}^{\infty} q^{n(n+1)/2} H_{n,q}(x) \frac{t^n}{[q]_n} = tq\, \frac{[-xtq^2]_\infty}{(-t^2;q^2)_\infty}, \tag{3.3}$$

so that

$$\sum_{n=0}^{\infty} q^{n(n+1)/2}\, \delta H_{n,q}(x) \frac{t^n}{[q]_n} =$$

$$tq \sum_{n=0}^{\infty} q^{n(n+1)/2} H_{n,q}(xq) \frac{t^n}{[q]_n},$$

which yields $\delta H_{0,q}(x) = 0$, and for $n \ge 1$,

$$\delta H_{n,q}(x) = (1-q^n)q^{1-n} H_{n-1,q}(xq) \tag{3.4}$$

or, more generally,

$$\delta^m H_{n,q}(x) = [q^{n-m-1}]_m\, q^{m(m-n)}\, H_{n-m,q}(xq^m), \quad n \ge m \ge 0.$$

Alternatively, we may write (3.3) as

$$\delta \sum_{n=0}^{\infty} q^{n(n+1)/2} H_{n,q}(x)(1+t^2) \frac{t^n}{[q]_n}$$

$$= tq \sum_{n=0}^{\infty} q^{n(n+1)/2} H_{n,q}(x) \frac{t^n q^n}{[q]_n},$$

which readily yields,

$$\delta H_{n,q}(x) + q^{1-2n} (1-q^{n-1}) (1-q^n) \delta H_{n-2,q}(x) =$$

$$(1-q^n)H_{n-1,q}(x).$$

It is easy to verify that

$$(x - \frac{t}{q})\delta \left\{ \frac{[-xtq]_\infty}{(-t^2;q^2)_\infty} \right\} - t\, \delta_t \left\{ \frac{[-xtq]_\infty}{(-t^2;q^2)_\infty} \right\} = 0,$$

which, in conjuntion with (3.2), gives

$$xq^{1+n} \delta H_{n,q}(x) = q^{1+n} (1-q^n)H_{n,q}(x) + (1-q^n) \delta H_{n-1,q}(x) \tag{3.5}$$

Using (3.4) in (3.5), we have

$$q^n H_{n,q}(x) - xqH_{n-1,q}(xq) + q^{1-n}(1-q^{n-1})H_{n-2,q}(xq) = 0. \tag{3.6}$$

From (3.4), in view of definition (1.1), we get

$$q^n H_{n,q}(x) = q^n H_{n,q}(xq) + xq(1-q^n) H_{n-1,q}(xq). \tag{3.7}$$

Notice, however, that by eliminating the term $q^n H_{n,q}(x)$ between (3.6) and (3.7), and then changing x to x/q, we at once obtain the pure recurrence relation

$$H_{n,q}(x) - xH_{n-1,q}(x) + q^{1-2n}(1-q^{n-1})H_{n-2,q}(x) = 0.$$

Further, in (3.5) replacing x by xq and appealing to (3.4), we get a q-difference equation for q-Hermite polynomials in the form:

$$\delta^2 H_{n,q}(x) - xq^3 \delta H_{n,q}(xq) + q^3(1-q^n)H_{n,q}(xq) = 0. \tag{3.8}$$

Following Moak [2,p.34], on account of (3.8), it is not hard to establish

$$\delta^2 u(x) + \left\{ \frac{1+q}{qx^2} - \frac{(1+q+x^2q^{4+n})(-x^2q^6;q^4)_\infty}{qx^2(-x^2q^4;q^4)_\infty} \right\} u(xq) = 0,$$

where $u(x) = H_{n,q}(x)/(-x^2q^4;q^4)_\infty$

Letting $b \to 0$ in the q-extension of Euler's transformation [3,p.348, eq.(281)], we obtain

$${}_2\phi_1 \left[\begin{matrix} 0,a;q,z \\ c \end{matrix} \right] = \frac{1}{[z]_\infty} {}_1\phi_1 \left[\begin{matrix} c/a;\ q,\ -az \\ c \end{matrix} \right], \tag{3.9}$$

which is a q-extension of Kummer's first formula [3,p.322,eq.(183)].

Since $\frac{1}{[x]_\infty} = \sum_{n=0}^{\infty} \frac{x^n}{[q]_n}$, we have

$$\delta^{2n} \left\{ \frac{1}{(-x^2q^2;q^2)_\infty} \right\} = \sum_{r=n}^{\infty} \frac{(-1)^r q^{2r}}{(q^2;q^2)_r} x^{2r-2n} [q^{2r-2n+1}]_{2n},$$

which on replacing r by n+r and doing some straightforward manipulations reduce to

$$\delta^{2n} \left\{ \frac{1}{(-x^2q^2;q^2)_\infty} \right\} =$$

$$(-1)^n q^{2n}(q;q^2)_n \; {}_2\phi_1 \left[\begin{matrix} 0,q^{1+2n};q^2,-x^2q^2 \\ q \end{matrix} \right] \tag{3.10}$$

Now, by applying (3.9) to (3.10) and then using (1.4) and (1.6), we find that

$$H_{2n,q}(x) = (-1)^{2n} q^{-3n-2n^2} (-x^2q^2;q^2)_\infty \delta^{2n} \left\{ \frac{1}{(-x^2q^2;q^2)_\infty} \right\} \tag{3.11}$$

Similarly, we can obtain

$$H_{2n+1,q}(x) =$$

$$(-1)^{2n+1}q^{-2-5n-2n^2}(-x^2q^2;q^2)_\infty \, \delta^{2n+1}\left\{\frac{1}{(-x^2q^2;q^2)_\infty}\right\} \quad (3.12)$$

Combining (3.11) and (3.12), we get a Rodrígues type representation for the polynomials $H_{n,q}(x)$ as

$$H_{n,q}(x) = (-1)^n q^{-n(n+3)/2} (-x^2q^2;q^2)_\infty \, \delta^n \left\{\frac{1}{(-x^2q^2;q^2)_\infty}\right\}$$

More generally, one can obtain

$$\delta^k \left\{\frac{H_{n,q}(x)}{(-x^2q^2;q^2)_\infty}\right\} =$$

$$(-1)^k q^{nk+(1/2)k(3+k)} \frac{H_{n+k,q}(x)}{(-x^2q^2;q^2)_\infty},$$

which, for k=1, reduces to a recurrence relation

$$xq^{n+2}H_{n+1,q}(x) + H_{n,q}(x) - (1+x^2q^2)H_{n,q}(xq) = 0.$$

One notes that all the properties for the polynomials $H_{n,q}(x)$ reduce, in view of (1.3), to corresponding properties for the Hermite polynomials $H_n(x)$.

REFERENCES

[1] HAHN, W.: "Uber orthogonalpolynome,die q-differenzengleichungen genügen". Math.Nachr. 2(1949), 4-34.

[2] MOAK, D.S.: "The q-analogue of the Laguerre polynomials". J. Math. Anal. Appl. 81(1981), 20-47.

[3] SRIVASTAVA, H.M. and KARLSSON, P.W.: "Multiple Gaussian Hypergeometric Series", Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York (1985).

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