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HANKEL TRANSFORM AND SERIES REPRESENTATION IN LAGUERRE POLYNOMIALS

L. I. Boyadjiev
Institute for Applied Mathematics
and Computer Science
High Institute for Machine and
Electrical Engineering,
P.O. Box 384
1000 Sofía, Bulgaria

ABSTRACT

In this paper we establish some theorems analogues to Rusev's results. We also study the character of the correspondence between the functional space $L(\lambda_0)$ and $G(\lambda_0)$ provided by the Hankel transform.

RESUMEN

En este trabajo se dan algunos teoremas análogos a los resultados de Rusev. Además se estudia la correspondencia entre el espacio funcional $L(\lambda_0)$ y $G(\lambda_0)$ generado por la transformada de Hankel.

1. INTRODUCTION

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers such that $-\limsup_{n\to\infty}(2\sqrt{n})^{-1}\ln|a_n| = \lambda_0 (0<\lambda_0 \leqq +\infty)$. It is well known |1| that in this case the region of convergence of a series in Laguerre polynomials

$$\sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(z) \tag{1}$$

is the interior $\Delta(\lambda_0)$ of the parabola $p(\lambda_0)$ defined by the equality $\mathrm{Re}\{(-z)^{1/2}\} = \lambda_0$. More precisely, the series (1) converges absolutely and uniformly on every compact subset of $\Delta(\lambda_0)$ and hence defines an holomorphic function $f(z)$ there.

The problem of representing analytic functions by series in Hermite polynomials $\{H_n(z)\}_{n=0}^{\infty}$ was solved by Hille [2] in 1940. In 1947 Pollard [3] published a paper in which, using Hille's results, he solved the problem of representing analytic functions by series in Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ for $\alpha=0$. Pollard's result was generalized by Rusev [4]. We mention here Rusev's result which will be used in our analysis. Let's define $L(\lambda_0)(0<\lambda_0 \leqq +\infty)$ to be the vector space of all complex functions f, holomorphic in the region $\Delta(\lambda_0)$ and having the following property: for every $0 \leqq \lambda < \lambda_0$ there exists a constant $D = D(f;\lambda)$ such that if $z = x + iy$ $\overline{\Delta(\lambda)}$: $\mathrm{Re}\{(-z)^{1/2}\} \leqq \lambda$, then the inequality $|f(z)| \leqq D \exp\{\Phi(\lambda;x,y)\}$ hols, where

$$\Phi(\lambda;x,y) = \frac{\sqrt{x^2+y^2+x}}{4} - \left\{\frac{x^2+y^2+x}{2}\left(\lambda^2 - \frac{x^2+y^2-x}{2}\right)\right\}^{1/2} \tag{2}$$

We have the following result due to Rusev [4]:

THEOREM I. Let $\alpha \neq -1,-2,\ldots$ be real and $0<\lambda_0 \leqq +\infty$. A complex function f holomorphic in the region $\Delta(\lambda_0)$ has a series representation there in Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ iff $f \in L(\lambda_0)$.

The integral representation of Laguerre polynomials [6, v.II, 10.12,(21)] allows us to consider the analytic function $z^{\alpha/2}\exp(-z)L_n^{(\alpha)}(z)$ which is holomorphic in the region $C-(-\infty, 0)]$, as a Hankel type integral transform of the function $(n!)^{-1}z^{n+\alpha/2}\exp(-z)$ holomorphic in the same region. Due to this fact we have that if an analytic function has a series representation in Laguerre polynomials, then this function is a Hankel's transform of a suitable complex (analytic) function. In [4] Rusev introduced the class $G(\sigma)(-\infty<\sigma\leqq+\infty)$ of all entire functions $\Phi(w)$ having the property

$$\limsup_{|w|\to +\infty} (2\sqrt{|w|})^{-1} \left[\ln\left| \Phi(w) - |w|\right|\right] \leq -\sigma$$

and characterized by the following

LEMMA 1. An entire function

$$\Phi(w) = \sum_{n=0}^{\infty} \frac{a_n}{n!} w^n \qquad (3)$$

belongs to the class $G(\sigma)$ iff

$$\limsup_{n\to +\infty} (2\sqrt{n})^{-1} \ln|a_n| \leq -\sigma \qquad (4)$$

The class $G(\sigma)$, when $\sigma = \lambda_o$, plays an important role in characterizing the class $L(\lambda_o)$, because the following statement holds [4]:

THEOREM 2. Let $0 < \lambda_o \leq +\infty$ and $\alpha > -1$. For a complex function f, holomorphic in the region $\Delta(\lambda_o)$, to be expanded there in a series in Laguerre polynomials, it is necessary and sufficient that the following representation holds in the region

$$\tilde{\Delta}(\lambda_o) = \Delta(\lambda_o) - (-\lambda_o^2, 0] \quad :$$

$$f(z) = z^{-\alpha/2} \exp z \int_0^{\infty} t^{\alpha/2} \exp(-t)\, \Phi(t) J_\alpha(2\sqrt{zt})\,dt, \qquad (5)$$

where the function $\Phi \in G(\lambda_o)$.

Let, as usual, $h_f(\theta)$ be an indicator function [7] of f. As a direct application of Theorem 2, Rusev proved that the following statement holds [5, p.40, (X)]

THEOREM 3. If f is an entire function of an exponential type less than one and $h_f(0) < 1/2$, then $f \in L(+\infty)$.

In this paper we shall prove that a result analogues to Theorem 2 holds for the functions of the class $G(\lambda_o)$ $(0 < \lambda_o \leq +\infty)$. We will also show that if the function (3) is of an exponential type less than one, then the corresponding function f from Theorem 2 belongs to the class $L(+\infty)$ and the assumption $h_f(0) < 1/2$ from Theorem 3 is fulfilled. Finally we study the caracter of the correspondence between the functional spaces $L(\lambda_o)$ and $G(\lambda_o)$ provided by the Hankel transform (5).

2. ENTIRE FUNCTIONS AND HANKEL TRANSFORM

On the basis of Theorem 2 one may suppose that every function of the class $G(\lambda_o)$ is an inverse Hankel's transform of a function of the class $L(\lambda_o)$. This turns out to be true and the following statement can be proved.

THEOREM 4. Let $0 < \lambda_o \leq +\infty$ and $\alpha > -1$. The entire function (3) is of the class $G(\lambda_o)$ iff the following representation holds in the region

$$\tilde{\Delta}(+\infty) = C - (-\infty, 0] \quad :$$

$$\Phi(t) = t^{-\alpha/2} \exp t \int_0^{\infty} x^{\alpha/2} \exp(-x) f(x) J_\alpha(2\sqrt{xt})\,dx, \qquad (6)$$

where the function f $L(\lambda_o)$.

Proof: From (2) and the asymptotic formula of Bessel's function [6, v.II, 7.13.1] it follows immediately that the integral in (6) is absolutely and uniformly convergent on every compact subset $K \subset \tilde{\Delta}(+\infty)$.

Let $f \in L(\lambda_o)$ and the representation (6) holds. From Theorem 1 it follows that the inequality (4) at $\sigma = \lambda_o$ is valid. We define

$$R_v(t) = \Phi(t) - \sum_{n=o}^{v} \frac{a_n}{n!} t^n$$

for $v = 0, 1, 2, \ldots$ and t $\tilde{\Delta}(+\infty)$. It is easy to see that

$$R_\nu(t) = \int_0^\infty x^\alpha \exp(-x) f(x) \left\{ \sum_{n=\nu+1}^{\infty} \frac{L_n^{(\alpha)}(x)}{\Gamma(n+\alpha+1)} t^n \right\} dx \tag{7}$$

For a fixed $t \in \tilde{\Delta}(+\infty)$, from the absolutely uniform convergence of the integral in (6) it follows that for every positive $\varepsilon > 0$ there exists $T = T(\varepsilon) > 0$ such that

$$\int_T^\infty x^\alpha \exp(-x) \, |f(x)| \left\{ \sum_{n=0}^{\infty} \frac{|L_n^{(\alpha)}(x)|}{\Gamma(n+\alpha+1)} |t|^n \right\} dx < \varepsilon$$

Then for every $\;= 0,1,2,\ldots$

$$\left| \int_T^\infty x^\alpha \exp(-x) f(x) \left\{ \sum_{n=\nu+1}^{\infty} \frac{L_n^{(\alpha)}(x)}{\Gamma(n+\alpha+1)} t^n \right\} dx \right| < \varepsilon \tag{8}$$

Further, there exists $N = N(\varepsilon) > 0$ with the property that if $\nu > N$ and $0 \leq x \leq T$, then

$$\left| \sum_{n=\nu+1}^{\infty} \frac{L_n^{(\alpha)}(x)}{\Gamma(n+\alpha+1)} t^n \right| < \varepsilon$$

Therefore

$$\left| \int_0^T x^\alpha \exp(-x) f(x) \left\{ \sum_{n=\;+1}^{\infty} \frac{L_n^{(\alpha)}(x)}{\Gamma(n+\alpha+1)} t^n \right\} dx \right| \tag{9}$$

$$= 0\left[\varepsilon \int_0^T x^\alpha \exp(-x) \, |f(x)| \, dx \right] = O(\varepsilon)$$

From (7), (8) and (9) we get that $R_\nu(t) = O(\varepsilon)$ $(\nu > N)$, i.e. the series (3) represents the function Φ in the region $\tilde{\Delta}(+\infty)$ and therefore in $\Delta(+\infty) = C$. Now from Lemma 1 it immediately follows that $\Phi \in G(\lambda_o)$.

Let us suppose now that the function (3) belongs to the class $G(\lambda_o)$. From Lemma 1 it follows that $\limsup_{n\to+\infty} (2\sqrt{n})^{-1} n \, |a_n| \leq \lambda_o$, i.e. the series $\sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(z)$

is absolutely uniformly convergent on every compact subset of $\Delta(\lambda_o)$ and therefore define there an analytic function of the class $L(\lambda_o)$. Let us define

$$f(z) = \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(z), \quad z \in \Delta(\lambda_o)$$

But, as we have just seen, the integral

$$t^{-\alpha/2} \exp t \int_0^\infty x^{\alpha/2} \exp(-x) f(x) \, J_\alpha(2\sqrt{xt}) dx$$

defines an entire function $\Phi^*(t)$ of the class $G(\lambda_o)$, when $t \in \tilde{\Delta}(+\infty)$ and

$$\Phi^*(t) = \sum_{n=0}^{\infty} \frac{a_n}{n!} t^n .$$

Therefore, $\Phi^* = \Phi$ and the representation (6) holds.

3. ENTIRE FUNCTIONS AND SERIES IN LAGUERRE POLYNOMIALS

Eigenfunction expansions are sometimes useful in studying the properties of the functions to which they converge. Here we consider entire functions and their expansions in terms of Laguerre series. We shall show that every function of this kind is an entire function of an exponential type which indicator function satisfies the inequality $h_f(0) < 1/2$, on the assumption that the functions define by (6) are entire functions of exponential type less than one. In fact, the following theorem holds.

THEOREM 5. Let (3) be an entire function of an exponential type less than one and $f(z)$ be defined by (5). Then $f(z)$ is an entire function of exponential type which indicator function satisfies the inequality $h_f(0) < 1/2$.

Proof: It is well known [7] that the type of Φ is given by the formula $\tau = \limsup_{n\to+\infty} \sqrt[n]{|a_n|}$. From the assumption that $\tau<1$ it follows that $-\limsup_{n\to+\infty} (2\sqrt{n})^{-1} \ln|a_n| = +\infty$, i.e. $\Phi \in G(+\infty)$. From Theorem 2 we conclude that $f \in L(+\infty)$, i.e. f is an entire function. Polya's representation enables us to conclude that the following equality holds

$$f(z) = \frac{1}{2\pi i} z^{-\alpha/2} \exp z \int_{\Gamma} K(z;\zeta) B_{\phi}(\zeta)\, d\zeta \quad \text{, where}$$

$$K(z;\zeta) = \int_0^{\infty} t^{\alpha/2} \exp\{-(1-\zeta)t\}\, J_{\alpha}(2\sqrt{zt})\, dt \qquad (10)$$

The contour Γ is any circular path with center at the origin and radius $\rho : \tau < \rho < 1$. $B_\phi(\zeta)$ is the Borel transform of Φ. It can easily be seen that, if $\zeta \in \Gamma$,

$$K(z;\zeta) = z^{\alpha/2} \exp(-z)(1-\zeta)^{-1-\alpha} \exp\left(-\frac{z\zeta}{1-\zeta}\right) \qquad (11)$$

From (10) and (11) we find the integral representation

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} (1-\zeta)^{-1-\alpha} \exp\left(-\frac{z\zeta}{1-\zeta}\right) B_{\phi}(\zeta)\, d\zeta \qquad (12)$$

If we put now

$$M(\rho) = \max_{\zeta\in\Gamma} |(1-\zeta)^{-1-\alpha} B_{\phi}(\zeta)| ,$$

$$\sigma(\rho) = \max_{\zeta\in\Gamma} |\zeta(1-\zeta)^{-1}| ,$$

then from (12) one may conclude that

$$|f(z)| \leqq \rho M(\rho) \exp\left[\sigma(\rho)\ |z|\right]$$

from where it can be seen that f is of exponential type.

The indicator function of f is defined by the formula [7]:

$$h_f(\theta) = \limsup_{\zeta\to+\infty} \zeta^{-1} \ln|f(\zeta \exp i\theta)|$$

In our case, having in mind (12), we find that

$$h_f(0) \leqq \max_{\zeta\in\Gamma} \operatorname{Re}\{\zeta(\zeta-1)^{-1}\} = \rho(1+\rho)^{-1} < 1/2$$

and so the theorem is proved.

4. ON AN ISOMORPHISM PROVIDED BY HANKEL TRANSFORM

From the validity of Theorem 2 and Theorem 4 we are encouraged to study more precizely the correspondence betwen the functional spaces $L(\lambda_o)$ and $G(\lambda_o)$ provided by the Hankel transform

$$H_{\alpha}(\gamma;z) = z^{-\alpha/2} \exp z \int_0^{\infty} t^{\alpha/2} \exp(-t)\Phi(t) J_{\alpha}(2\sqrt{zt})\,dt \qquad (13)$$

where J_α is the Bessel function of the first kind with index α. First of all we prove the following two preliminary statements:

LEMMA 2. If $0< \lambda_o \leqq +\infty$ and $\alpha>-1$, then the Hankel transform (13) provid a linear one to one correspondance betwen $L(\lambda_o)$ and $G(\lambda_o)$.

Proof: If $f_1(z)$ and $f_2(z)$ are an arbitrary functions belonging to $L(\lambda_o)$, from [4, Theorem 4.3, (b)] follows that $\lambda f_1(z) + \mu f_2(z)$ belongs to $L(\lambda_o)$, where λ and μ are arbitrary complex numbers. In the same manner it can be proved that $G(\lambda_o)$ is a linear functional space too.

From the linear property of the Hankel's transform (13) it is clear that if Φ_1 and Φ_2 belong to $G(\lambda_o)$ then

$$H_{\alpha}(\lambda\Phi_1 + \mu\Phi_2;z) = \lambda H_{\alpha}(\Phi_1;z) + \mu H_{\alpha}(\Phi_2;z)$$

as well as that

$$H_{\alpha}^{-1}(\lambda f_1 + \mu f_2;w) = \lambda H_{\alpha}^{-1}(f_1;w) + \mu H_{\alpha}^{-1}(f_2;w)$$

Let f(z) can be represented in the region $\Delta(\lambda_o)$ by a series of the kind (1). From the generating function formula for Laguerre polynomials [6, 10.12,(18)] one get

$$(n!)^{-1} w^n = w^{-\alpha/2} \exp w \int_0^{\infty} x^{\alpha/2} \exp(-x) L_n^{(\alpha)}(x) J_{\alpha}(2\sqrt{wx})\,dx$$

Then for $w \in C$,

$$\Phi(w) = H_{\alpha}^{-1}(f;w) =$$

$$\sum_{n=0}^{\infty} a_n \left\{ w^{-\alpha/2} \exp w \int_0^{\infty} x^{\alpha/2} \exp(-x) L_n^{(\alpha)}(x)\, J_{\alpha}(2\sqrt{wx})\, dx \right\} =$$

$$\sum_{n=0}^{\infty} \frac{a_n}{n!} w^n$$

Having in mind that $-\limsup_{n\to+\infty}(2\sqrt{n})^{-1}\ell n\,|a_n| \geqq \lambda_o$ [4, Theorem 4.3,(b)] we can make the conclusion that $\Phi(w)\in G(\lambda_o)$.

Let Φ is an arbitrary function of $G(\lambda_o)$. From the formula [6,10.12,(21)] for Laguerre polynomials it follows that for $z \in \Delta(\lambda_o)$,

$$f(z) = H_{\alpha}(\Phi;z) = \sum_{n=0}^{\infty} \frac{a_n}{n!}$$

$$\left\{ z^{-\alpha/2} \exp z \int_0^{\infty} t^{n+\alpha/2} \exp(-t) J_{\alpha}(2\sqrt{zt})\, dt \right\} =$$

$$\sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(z)$$

Because $-\limsup_{n\to+\infty}(2\sqrt{n})^{-1}\ell n|a_n| \geqq \lambda_o$ from [4, Theorem 4.3, (b)] it follows that $f(z)\in L(\lambda_o)$.

Finally from Theorem 2 and Theorem 4 follows that this Lemma is valid.

If we denote the differential operator

$\dfrac{\partial^k}{\partial w^k}$ by $D_w^{(k)}$,

the following statement can be proved:

LEMMA 3. If $0<\lambda_o \leqq +\infty$, $\alpha>-1$ and the function $f(z)$ belongs to $L(\lambda_o)$, then for an arbitrary integer $k\geqq 0$ the integral

$$\int_0^{\infty} x^{\alpha/2} \exp(-x) f(x)\, D_w^{(k)} \left[w^{-\alpha/2} \exp w J_{\alpha}(2\sqrt{wx}) \right] dx \tag{14}$$

defines a function belonging to $G(\lambda_o)$

Proof: By induction, it can be easyly proved using the formula [6,10.12,(18)], that for an arbitrary integer $k\geqq 0$,

$$D_w^{(k)}\left[w^{-\alpha/2}\exp w J_{\alpha}(2\sqrt{wx})\right] =$$

$$\sum_{m=o}^{k} (-1)^m \binom{k}{m} x^{m/2} w^{-(m+\alpha)/2} \exp w\, J_{\alpha+m}(2\sqrt{wx})$$

Thus, the integral (14) can be represented as

$$\exp w \sum_{m=o}^{k} (-1)^m \binom{k}{m} w^{-(m+\alpha)/2}$$

$$\left\{ \int_0^{\infty} x^{(m+\alpha)/2} \exp(-x) f(x) J_{\alpha+m}(2\sqrt{wx})\, dx \right\} \tag{15}$$

Let $M\subset C$ be an arbitrary compact set and

$$\lambda^* = \max_{w\in M} \operatorname{Re}(-w)^{1/2}$$

From Theorem 1 and the asymptotic formula for Bessel functions [6,7.13.1,(3)] it follows, that for an arbitrary $\delta>0$, $m=0,1,2,\dots,k$ and $w\in M$

$$|x^{(m+\alpha)/2}\exp(-x) f(x) J_{\alpha+m}(2\sqrt{wx})|$$

$$= O\left\{ x^{(m+\alpha)/2} \exp(-x/2) \exp(2\lambda^*-\lambda_o+\delta)\sqrt{x} \right\}$$

Therefore the integrals in (15) absolutely uniformly converge on M, i.e. the integral (14) defines an entire function.

Let we assume that the function $f(z)$ has a representation by a series (1), where the coefficients are given by the formula

$$a_n = \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} \int_0^{\infty} t^{\alpha} \exp(-t) f(t) L_n^{(\alpha)}(t)\, dt,$$

$$n = 0,1,2,\dots$$

From [6,10.12,(18)] it follows, that for an arbitrary integer $k\geqq 0$

$$\int_0^\infty x^{\alpha/2}\exp(-x)f(x)D_w^{(k)}\left[w^{-\alpha/2}\exp w\, J_\alpha(2\sqrt{wx})\right]dx =$$

$$\sum_{n=0}^{\infty}\frac{(n+k)!}{n!(n+k+\alpha)!}\left\{\int_0^\infty x^\alpha\exp(-x)f(x)L_{n+k}^{(\alpha)}(x)\,dx\right\}w^n =$$

$$\sum_{n=o}^{\infty}\frac{a_{n+k}}{n!}w^n$$

According to [4, Theorem 4.3.(b)], for an arbitrary integer $k\geq 0$, $\limsup_{n\to+\infty}(2\sqrt{n})^{-1}\ln|a_{n+k}|\leq -\lambda_o$ and therefore

$$\sum_{n=o}^{\infty}\frac{a_{n+k}}{n!}w^n\in G(\lambda_o).$$

Using the lemmas proved above we can prove now the following.

THEOREM. The Hankel transform (13) provides an isomorphism betwen the functional spaces $L(\lambda_o)$ and $G(\lambda_o)$

Proof: It's known [8,2.3] that $L(\lambda_o)$ and $G(\lambda_o)$ are multinormed linear spaces. Let us equip $G(\lambda_o)$ with a topology defined by the multinorm $\{\gamma_{M,k}\}_{k=o}^{\infty}$, where

$$\gamma_{M,k}(\Phi) = \sup_{w\in M}|\Phi^{(k)}(w)|, \quad \Phi\in G(\lambda_o)$$

and $M\subset C$ is an arbitrary compact set. If for every compact subset $k\subset\Delta(\lambda_o)$ we define a seminorm $\eta_k(f)$ on $L(\lambda_o)$ by the formula $\eta_k(f)=\sup_{z\in k}|f(z)|, f\ L(\lambda_o)$, then the topology of this space can be defined by the multinorm $\{\eta_k\}$.

Let $\Phi(w)\in G(\lambda_o)$ is an arbitrary function. From Theorem 4 and Lemma 3 follows that

$$\Phi^{(k)}(w) = D_w^{(k)}(\Phi)\in G(\lambda_o)$$

and

$$|\Phi^{(k)}(x)| \leq \int_0^\infty x^{\alpha/2}\exp(-x)|f(x)||D_w^{(k)}$$

$$\left[w^{-\alpha/2}\exp w\, J_\alpha(e\sqrt{wx})\right]|\,dx$$

As the integral (14) converger absolutely and uniformly, there exists $T>0$ such that for $w\in M$ and $k\geq 0$

$$\int_T^\infty x^{\alpha/2}\exp(-x)\,|f(x)||D_w^{(k)}\left[w^{-\alpha/2}\exp w J_\alpha(2\sqrt{wx})\right]|dx<\varepsilon$$

where $\varepsilon>0$ is an arbitrary small number. In this way one gets that

$$\gamma_{M,k}(\Phi)\leq B_\alpha(w;T)\cdot\eta_{[0,T]}(f) \qquad (16)$$

where

$$B_\alpha(w;T) = \int_0^T x^{\alpha/2}\exp(-x)|D_w^{(k)}\left[w^{-\alpha/2}\exp w\, J_\alpha(2\sqrt{wx})\right]|dx$$

In the same way it can be seen that there exists $T_1>0$ such that for $z\in k$,

$$\eta_k(f)\leq A_\alpha(z;T_1)\cdot\gamma_{[0,T_1],o}(\Phi)$$

where

$$A_\alpha(z;T_1)=\int_0^{T_1} t^{\alpha/2}\exp(-t)\,|z^{-\alpha/2}\exp z\, J_\alpha(2\sqrt{zt})|\,dt$$

Let $\{\Phi_\nu\}_{\nu=o}^{\infty}$ be an arbitrary sequence of functions belonging to $G(\lambda_o)$, which converges to a function $\Phi\in G(\lambda_o)$, i.e. according [8, Lemma L6.1], for an arbitrary seminorm on $G(\lambda_o)$ and $k=0,1,2,..$ $\lim_{\nu\to\infty}\gamma_{M,k}(\Phi_\nu-\Phi)=0$. If we denote $f_\nu(z)=H_\alpha(\Phi_\nu;z)$ and $f(z)=H_\alpha(\Phi;z)$ from (17) and Lemma 2 it follows that for an arbitrary seminorm on $L(\lambda_o)$, $\lim_{\nu\to\infty}\eta_k(f_\nu-f)=0$.

Let now $\{f_\nu\}_{\nu=o}^{\infty}$ be an arbitrary sequence of functions belonging to $L(\lambda_o)$, which converges to a function $f\in L(\lambda_o)$. Then for an arbitrary seminorm on $L(\lambda_o)$, $\lim_{\nu\to+}\eta_k(f_\nu-f)=0$. If we denote $\Phi_\nu(w)=H_\alpha^{-1}(f_\nu;w)$ and $\Phi(w)=H_\alpha^{-1}(f;w)$, then from (16) and Lemma 2 follows that for an arbitrary seminorm on $G(\lambda_o)$, $\lim_{\nu\to+\infty}\gamma_{M,k}(\Phi_\nu-\Phi)=0$. According to [8, Lemma 1.6.1] we conclude that the sequence $\{H_\alpha^{-1}(f_\nu;w)\}$ converges to $H_\alpha^{-1}(f;w)$.

REFERENCES

[1] SZEGO, G.: *"Orthogonal polynomials"*, New York, 1959.

[2] HILLE, E.: (I) Duke Math. Journ. 5, 1939, 875-936. (II) Trans. Amer. Math. Soc., 47, 1940,80-94.

[3] POLLARD, H.: Ann.Math. (2), 48, 1947, 358-365.

[4] RUSEV, P.: *"Analytic Functions and Classical Orthogonal Polyn mials"*, Sofia, 1984.

[5] BOAS, R.P.; BUCK, R.C.: *"Polynomial expansions of analytic functions"*. Ergebn. Math. Grenzgeb. N. Folge, 19, 1958.

[6] BATEMAN, H.; ERDELYI, A.: *"Higher transcendental functions"*. New York, 1953.

[7] BOAS, R.P.: *"Entire functions"*, New York, 1954.

[8] ZEMANIAN, A.H.: *"Generalized integral transformations"*, New York, 1968.

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