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ON RATE OF CONVERGENCE OF FOURIER SERIES OF A FUNCTION OF WIENER'S CLASS

R. N. Siddiqi
Department of Mathematics
Kuwait University
P.O. Box 5969
13060 Safat,
Kuwait

ABSTRACT

The main object of this paper is to obtain the rate of convergence of Fourier series of a function of Wiener's class. From our theorem we can deduce a generalized version of a theorem of Wiener.

RESUMEN

El objeto principal de este trabajo es conseguir la rata de convergencia de serie de Fourier de funciones de clase Wiener. Se da una versión generalizada del teorema de Wiener.

1. INTRODUCTION

Let f be a 2π-periodic function defined on $[0,2\pi]$ We set

$$\overset{b}{\underset{a}{V_p}}(f) = \sup\left\{ \sum_{i=1}^{n} [f(t_i) - f(t_{i-1})]^p \right\}^{1/p} \quad (1 < p < \infty),$$

where suprema has been taken with respect to all partitions P : $a = t_0 < t_1 < t_2 < \dots\ t_n = b$ of any segment $[a,b]$ contained in $[0,2\pi]$. We call $V_p(f)$ the p-th total variation of f on $[a,b]$. If we denote p-th total variation of f on $[0,2\pi]$ by $V_p(f)$, then we can define Wiener's class simply by

$$V_p = \{ f : V_p(f) < \infty \} . \tag{1}$$

It is clear that V_1 is an ordinary class of functions of bounded variation, introduced by Jordan. The class V_p was first introduced by N. Wiener [2]. He [2] showed that the functions of class V_p could only have simple discontinuities. We note [3] that

$$V_{p_1} \subset V_{p_2} \qquad (1 \leq p_1 < p_2 < \infty) \tag{2}$$

is a strict inclusion. Hence for an arbitrary $1 < p < \infty$, Wiener's class V_p is strictly larger than the class V_1.

2. MAIN RESULT

Let $f \in V_p (1 \leq p < \infty)$ and let

$$S(f) = \frac{1}{2} a_o + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be its Fourier series. Wiener [2] proved the following theorem

THEOREM A. If $f \in V_p (1 < p < \infty)$ then S(f) converges almost everywhere in $[0,2\pi]$.

Recently R. Bojanic [1] gave an estimate of the rate convergence of Fourier series of functions of bounded variation in the following form:

THEOREM B. If $f \in V_1$, then

$$|S_n(x) - 1/2 \ \{f(x+0) + f(x-0)\}| \leq \frac{3}{n} \sum_{k=1}^{n} \overset{\pi/k}{\underset{o}{V_1}}(g_x(t))$$

where $\overset{k}{\underset{o}{V_1}}(g_x(t))$ is the first total variation of

$$g_x(t) = f(x+t) + f(x-t) - f(x+0) - f(x-0) \tag{3}$$

and $S_n(x)$ is the n-th partial sum of S(f).

The main object of this paper is to obtain the rate of convergence of Fourier series of $f \in V_p$ $(1 < p < \infty)$ which is strictly larger class than the class V_1. To be precise, we prove the following theorem.

THEOREM 1. If $f \in V_p (1 < p < \infty)$, then

$$|S_n(x) - 1/2\ \{f(x+0) + f(x-0)\}| \le \frac{3M}{n} \sum_{k=1}^{n} V_{0p}^{\pi/k}(g_x(t))$$

for $|x| \le \pi$ where M is a positive real number.

3. PROOF

Since we can write

$$S_n(x) - 1/2\ \{f(x+0) + f(x-0)\} =$$

$$\pi^{-1} \int_0^{\pi} \frac{\sin nt}{t}\, g_x(t)dt + O(1) =$$

$$\pi^{-1} \sum_{k=0}^{n-1} \int_{\frac{k\pi}{n}}^{\frac{(k+1)\pi}{n}} \frac{\sin nt}{t}\, g_x(t)dt + O(1)\ .$$

By change of variable the above expression can be written into

$$= (\pi)^{-1} \sum_{k=0}^{n} \int_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}}$$

$$\left(\frac{g_x(t+2k\pi/n)}{t+2k\pi/n} - \frac{g_x(t + (2k+1)(\pi/n))}{t + (2k+1)\ (\pi/n)}\right) \sin nt\ dt + O(1)$$

$$= (\pi)^{-1} \int_0^{\frac{\pi}{n}} \left[\sum_{k=1}^{[n/2]}\right.$$

$$\left.\left(\frac{g_x(t+2k\ /n)}{t+2k\pi/n} - \frac{g_x(t+(2k+1)(\pi/n))}{t+\ (2k+1)\ (\pi/n)}\right)\right] \sin nt + O(1)$$

$$= (\pi)^{-1} \int_0^{\frac{\pi}{n}} \left[\sum_{k=1}^{[\varepsilon n]} + \sum_{[\varepsilon n]+1}^{[n/2]}\right] \sin nt\ dt + O(1)$$

$$= I_n(\varepsilon) + J_n(\varepsilon) + O(1) \qquad (4)$$

Now we consider

$$I_n(\varepsilon) = (\pi)^{-1} \int_0^{\frac{\pi}{n}} \sum_{k=1}^{[\varepsilon n]}$$

$$\left[\left(\frac{g_x(t+2k\pi/n)}{t+2k\pi/n} - \frac{g_x(t+(2k+1)(\pi/n))}{t+(2k+1)(\pi/n)}\right)\right] \sin nt\ dt$$

$$\le (\pi)^{-1} \int_0^{\pi/n}$$

$$\left.\frac{g_x(t+2k\pi/n) - g_x(t+(2k+1)(\pi/n))}{t+2k\pi/n}\right] \sin nt\ dt +$$

$$+ \frac{1}{n} \int^{\pi/n} \left[\sum_{k=1}^{[\varepsilon n]} \frac{g_x(t+(2k+1)(\pi/n))}{(t+2k\ /n)(t+2k+1)(\pi/n)}\right] \sin nt\ dt$$

$$= I_{n_1}(\varepsilon) + I_{n_2}(\varepsilon) \qquad (5)$$

Applying Hölder's inequality on the sum of integrand, we obtain,

$$I_{n_1}(\varepsilon) \le (\pi)^{-1}$$

$$\left[\int_0^{\pi/n} \left|\sum_{k=1}^{[\varepsilon n]} \right| g_x(t+2k\pi/n) - g_x(t+(2k+1)(\pi/n))|^p\right]^{1/p} \times$$

$$\left(\sum_{k=1}^{|\varepsilon n|} \left|\frac{1}{2k\pi/n}\right|^q\right)^{1/q} \sin nt\ dt \qquad (6)$$

$$\le \frac{M_1}{n} 3\varepsilon\pi\ V_{0p}(g_x(t)) \qquad (7)$$

because the series of right hand side of (6) converges for $q > 1$, hence we can find a positive number M_1 satisfying the above inequality (7).
And also

$$I_{n_2}(\varepsilon) =$$

$$\frac{1}{n}\int_0^{\pi/n} \left| \sum_{k=1}^{[\varepsilon n]} \frac{g_x(t+(2k+1)(\pi/n))}{(t+2k\pi/n)(t+(2k+1)(\pi/n))} \right| \sin nt\, dt$$

$$\frac{1}{n} \sup_{0\le t\le 3\varepsilon\pi} \{g_x(t)\} \le \frac{1}{n} \overset{3\varepsilon\pi}{\underset{o}{V_p}}(g_x(t))\, M_2 \tag{8}$$

where M_2 is a positive real number. Hence from (7) and (8), we obtain

$$I_n(\varepsilon) \le \frac{M_1}{n} \overset{3\varepsilon\pi}{\underset{o}{V_p}} (g_x(t)) + \frac{M_2}{n} \overset{3\varepsilon\pi}{\underset{o}{V_p}} g_x(\varepsilon)$$

$$\le \frac{M_3}{n} \overset{3\varepsilon\pi}{\underset{o}{V_p}} (g_x(t)) \tag{9}$$

where $M_3 = M_1 + M_2$. Similarly we can prove that

$$|J_n(\varepsilon)| \le \frac{1}{n} \overset{\pi}{\underset{o}{V_p}} (g_x(t)) + \frac{1}{n} \le \frac{2}{n} \overset{\pi}{\underset{o}{V_p}} (g_x(t)). \tag{10}$$

Collecting the terms of (9) and (10), we obtain

$$|S_n(x) - 1/2\,[f(x+0) + f(x-0)]|$$

$$\le \frac{M_3}{n} \overset{3\varepsilon\pi}{\underset{o}{V_p}} (g_x(t)) + \frac{2}{n} \overset{\pi}{\underset{o}{V_p}} (g_x(t))$$

$$\le \frac{3M}{n} \sum_{k=1}^{n} \overset{\pi/k}{\underset{o}{V_p}} (g_x(t))$$

where M is a positive real number. This completes the proof of our main theorem.

Since $\overset{t}{\underset{o}{V_p}} (g_x(t))$ is continuous when f(x) is continuous, it follows that

$$\frac{1}{n} \sum_{k=1}^{n} \overset{\pi/k}{\underset{o}{V_p}} (g_x(t)) \to 0 \quad \text{as} \quad n \to \infty.$$

Thus we deduce the following result which is generalized version of Theorem A due to Wiener [2] (cf. Zygmund [4] p.59).

COROLLARY. If $f \in V_p(1\le p<\infty)$ then the Fourier series of f converges to $1/2\ [f(x+0) + f(x-0)]$ at every $x \in [0,2\pi]$. In particular, S(f) converges to f(x) at every point x of continuity of f of V_p $(1\le p<\infty)$.

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