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RECURSION RELATIONS FOR A CLASS OF GENERALIZED ELLIPTIC INTEGRALS

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ABSTRACT

Our purpose in this note is to provide a number of recursion formulas for $R_\mu(k,\alpha,\gamma)$, so that the numerical results can be extended to arbitrary parameter values.

RESUMEN

El objeto de esta nota es proveer algunas formulas de recursión para $R_\mu(k,\alpha,\gamma)$, de manera que los resultados numéricos pueden ser extendidos a valores arbitrarios de los parámetros.

1. INTRODUCTION

Since the introduction of the functions

$$\Omega_\mu(k) = \int_0^\pi (1 - k^2\cos(\theta))^{-\mu-\frac{1}{2}} d\theta \quad (1)$$

nearly twenty years ago by Epstein and Hubbell [1], these integrals have attracted much attention due to the importance they have in radiation and other problems, and they have been generalized in the series of papers [2-6]. The specific integrals in these papers are all ultimately expressible in terms of the class [4]

$$R_\mu(k,\alpha,\gamma) = \int_0^\pi \frac{\cos^{2\alpha-1}(\theta/2)\sin^{2\gamma-2\alpha-1}(\theta/2)}{(1 - k^2\cos(\theta))^{\mu+1/2}} d\theta; \quad (2)$$

accordingly, we shall formulate our results in terms of this function. Kalla et al. [4] have given exact expressions for (2) in terms of the hypergeometric function and have provided a short table of numerical values. For practical usage it is important to have such a table supplemented by a set of recursion formulas so the numerical results can be extended to arbitrary parameter values. Our purpose in this note is to provide a family of such recursion formulas, which are presented in the following section. In part 3 we list a number of limiting cases of $R_\mu(k,\alpha,\gamma)$, which appear to have independent interest. We do not include the derivation of these results, but, because of the exact expresion in [4], they can all be reduced to properties of the hypergeometric function and can be verified accordingly.

2. RECURSION PROPIERTIES OF $R_\mu(k,\alpha,\gamma)$

Through integration by parts in (2), Kalla et al. [4] have shown that

$$R_\mu(k,\alpha,\gamma) = R_\mu(k,\alpha,\gamma-1) - R_\mu(k,\alpha+1,\gamma). \quad (3)$$

Since in this relation both α and γ are effected, we say that (3) is of type $(\alpha,\gamma)_1$. (The subscript numbers the various relations of this type). Accordingly, we have found the following recursion relations:

$$((\alpha)) \quad R_\mu(k,\alpha+1,\gamma) = \frac{(\alpha-1)(1-k^2)}{(\gamma-\alpha)(1+k^2)} R_\mu(k,\alpha-1,\gamma)$$

$$- \frac{[(\gamma-2\alpha)(1+k^2)-(2\gamma-2\alpha-2\mu-1)k^2]}{(\gamma-\alpha)(1+k^2)} R_\mu(k,\alpha,\gamma)$$

$((\mu))\quad R_{\mu+1}(k,\alpha,\gamma) = \frac{(\gamma-\mu-1/2)}{(\mu+1/2)(1-k^4)} R_{\mu-1}(k,\alpha,\gamma)$

$+ \frac{[(2\mu+1-\gamma)(1+k^2)-(2\mu-2\gamma+2\alpha+1)k^2]}{(\mu+1/2)(1-k^4)} R_\mu(k,\alpha,\gamma)$

$((\alpha,\gamma)_2)\quad R_\mu(k,\alpha,\gamma) = R_\mu(k,\alpha-1,\gamma-1)-R_\mu(k,\alpha-1,\gamma)$

$((\alpha,\gamma)_3)\quad R_\mu(k,\alpha+1,\gamma+1)=\frac{(\alpha-1)(\gamma-1)(1-k^2)}{\gamma(2\gamma-2\mu-1)k^2} \times$

$R_\mu(k,\alpha-1,\gamma-1)-\frac{[(\gamma-1)(1+k^2)-(2\gamma+2\alpha-2\mu-3)k^2]}{(2\gamma-2\mu-1)k^2}R_\mu(k,\alpha,\gamma)$

$((\mu,\alpha))\quad R_{\mu+1}(k,\alpha,\gamma)= \frac{\mu+1/2+\alpha-\gamma}{(\mu+1/2)(1+k^2)} R_\mu(k,\alpha,\gamma)$

$+ \frac{(\alpha-1)}{(\mu+1/2)(1+k^2)} R_\mu(k,\alpha-1,\gamma)$

$((\alpha,\gamma,\mu)_1)\quad R_\mu(k,\alpha,\gamma)=(1-k^2)^{\alpha-\mu-1/2} \times$

$(1+k^2)^{\gamma-\mu-\alpha-1/2} R_{\gamma-\mu-1}(k,\gamma-\alpha,\gamma)$

$((\alpha,\gamma,\mu)_2)\quad R_\mu(k,\alpha,\gamma) =$

$\frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)(1+k^2)^{\gamma-\alpha-\mu-1/2}}{\Gamma(\gamma-\mu-1/2)\Gamma(\mu+1/2)} R_{\gamma-\alpha-1/2}(k,\gamma-\mu-1/2,\gamma)$

$((\alpha,\gamma,\mu)_3[4])\quad R_\mu(k,\alpha,\gamma) = \frac{\alpha-1}{(1-2\mu)k^2}R_{\mu-1}(k,\alpha-1,\gamma-1)$

$- \frac{(\gamma-\alpha-1)}{(1-2\mu)k^2} R_{\mu-1}(k,\alpha,\gamma-1)$

$((\alpha,\gamma,\mu)_4[4])\quad R_\mu(k,\alpha,\gamma) =$

$(1+k^2)R_{\mu+1}(k,\alpha,\gamma)-2k^2R_{\mu+1}(k,\alpha+1,\gamma+1)$

$((\mu,\gamma)_1[4])\quad R_\mu(k,\alpha,\gamma) =$

$(1-k^2)R_{\mu+1}(k,\alpha,\gamma)+2k^2R_{\mu+1}(k,\alpha,\gamma+1)$

$((\alpha,\gamma,\mu)_5)\quad R_\mu(k,\alpha,\gamma) =$

$(1+k^2)^{-1}R_{\mu-1}(k,\alpha,\gamma)+ \frac{2k^2}{1+k^2} R_\mu(k,\alpha+1,\gamma+1)$

$((\mu,\gamma)_2[4])\quad R_\mu(k,\alpha,\gamma) =$

$(1-k^2)^{-1}R_{\mu-1}(k,\alpha,\gamma)- \frac{2k^2}{1-k^2} R_\mu(k,\alpha,\gamma+1)$

3. LIMITING CASES AND SPECIAL VALUES

$$R_\mu(k,\alpha,\mu+1/2) = \frac{\Gamma(\alpha)\Gamma(\mu-\alpha+1/2)}{\Gamma(\mu+1/2)}\frac{(1-k^2)^{\alpha-\mu-1/2}}{(1+k^2)^\alpha} \quad (3.1)$$

$$R_\mu(k,1/2-\mu,1/2) = \frac{\Gamma(\mu)\Gamma(1/2-\mu)}{2\sqrt{\pi}\sqrt{1+k^2}(1-k^2)^{2\mu}} \times \left\{(1+3k^2-\sqrt{8k^2(1+k^2)})^\mu+(1+3k^2+\sqrt{8k^2(1+k^2)})^\mu\right\} \quad (3.2)$$

$$R_\mu(k,\mu+1,2\mu+1) = \frac{2^{\mu-1}\Gamma^2(\mu)}{\Gamma(2\mu)}\frac{(1-\sqrt{1-k^4})^\mu}{k^{4\mu}\sqrt{1+k^2}} \quad (3.3)$$

$$R_\mu(k,\mu,2\mu) = \frac{2^{\mu-1}\Gamma^2(\mu)}{\Gamma(2\mu)(1+k^2)^{1-\mu}}\frac{\sqrt{1+k^2}-\sqrt{1-k^2}}{\sqrt{1+k^2}(1+k^4)^\mu} \quad (3.4)$$

$$R_\mu(3^{-1/2},\alpha,\gamma)=\sqrt{\pi}\frac{\Gamma(\gamma-\alpha)\Gamma(\alpha)}{\Gamma(\frac{\gamma-\alpha+1}{2})\Gamma(\frac{2\mu+3}{4})}(\frac{3}{4})^{\mu+1/2} \quad (3.5)$$

$$R_\mu(1,\alpha,\gamma) = \frac{\Gamma(\gamma-\alpha)\Gamma(\alpha-\mu-1/2)}{2^{\mu+1/2}\Gamma(\gamma-\mu-1/2)} \quad (3.6)$$

$$R_0(k,1/2,1)=\frac{2}{\sqrt{1+k^2}} K(\frac{2k^2}{1+k^2}) \quad (3.7)$$

$$R_{1/2}(k,1,2) = \frac{1}{2k^2} \ell n\,(\frac{1+k^2}{1-k^2}) \quad (3.8)$$

$$R_{1/2}(k,\ 1,\ 3/2) = R_0(k,1/2,3/2) = \frac{1+k^2}{4k^2}\ \ell n\left(\frac{1+3k^2}{1-k^2}\right) \tag{3.9}$$

4. EXAMPLE

Thus, if we require $R_{1/2}(.8,1.5,2)$, by using $(\alpha,\gamma,\mu)_1$, we obtain

$$R_{1/2}(k,1.5,2) = \sqrt{\frac{1-k^2}{1+k^2}}\ R_{1/2}(k,.5,2)$$

so from Table 1. of reference [4] we obtain the value 0.6111602. Through the use of the results given here, R need only be tabulated for

$$0\leq\mu,\alpha\leq 1\leq\gamma\leq 2$$

with $0\leq k\leq 1$.

REFERENCES

[1] EPSTEIN, L.F. and HUBBELL, J.H., J.Res. NBS 67, 1-17(1963).

[2] KALLA, S.L.: *"Mathematical Structures-Computational-Mathematics-Mathematical Modelling"*, 2, 216-219(1984).

[3] KALLA, S.L. and AL-SAQABI, B.: Revista Brasileira de Física, 16, 145-156(1986); 18, 135-155(1988).

[4] KALLA, S.L., CONDE, S. and HUBBELL, J.H.: Appl. Anal. 22, 273-287(1986).

[5] KALLA, S.L., LEUBNER, C. and HUBBELL,J.H.: Appl. Anal. 25, 269-274(1987).

[6] AL-SAQABI, B.N.: "*A Generalization of Elliptic type Integrals*". Hadronic Jour. 10,331-337(1987)

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