Rev. Téc. Ing., Univ. Zulia
Vol. 10, No. 2, 1987

ON SUMMATION OF A CLASS OF Q-Series

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ABSTRACT

In the present work, the author shows must of the results of Pandey and Saram, Sharma and Srivastava on generalizations of some classical hypergeometric summation theorems for multiple hypergeometric series, are special cases of author's previous results.

RESUMEN

En el presente trabajo, el autor muestra que la mayoría de los resultados de Pandey y Saram, Sharma y Srivastava sobre generalizaciones de algunos teoremas clásicos de suma de series hipergeométricas para series hipergeométricas múltiples, son casos especiales de resultados obtenidos por él.

&1. INTRODUCTION

In the recent past Sharma [3], [4] and Pandey and Saran [2] discussed the double series generalizations of the classical hypergeometric summation theorems of Saalschütz, Watson, Dixon, Whipple, Dougall and others. Very recently, Srivastava [6] tried to unify these summations of ordinary hypergeometric series. Here an attempt has been made to show that most of the above summations are deducible as very special cases of the certain recently communicated results due to the auther [1;(2.1) and (2.2)]. These results were discovered from partition theoretic point of view and the present aspect was somehow or other overlooked.

&2. DEFINITIONS AND NOTATIONS

We define a basic hypergeometric function as

$$\phi\begin{bmatrix}(a);x\\(b)\end{bmatrix} = \sum_{n\geq 0} \frac{[(a)]_n\, x^n}{[(b)]_n\,[q]_n}, \quad (|x|<1)$$

where the parameters (a) in small bracket stand for the sequence of A parameters $a_1, a_2, \ldots, a_A$ (similar interpretation applies for (b) also). Further, $[\alpha]_n$ stand for

$$(1-\alpha)(1-\alpha q)(1-\alpha q^2)\ldots(1-\alpha q^{n-1}),\ [\alpha]_o = 1,\ |q|<1 .$$

Whenever α is an integer, say **m**, we denote it by $[q^m]_n$ instead of $[m]_n$.

In what follows, the other notations and definitions carry their usual meaning.

We shall make use of the following results due to the author [1; (2.1) and (2.2)] to deduce the required summations,

$$\sum_{m_1,\ldots,m_n\geq o} \frac{[(a)]_{M_1}[\alpha]_{M_2}[\alpha b_1]_{M_3}[\alpha b_1 b_2]_{M_4}\ldots}{[(c)]_{M_1}[\alpha b_1 b_2]_{M_2}[\alpha b_1 b_2 b_3]_{M_3}\ldots} \times \tag{1}$$

$$\times \frac{\ldots[\alpha b_1 b_2\ldots b_{n-2}]_{M_n}\ [b_1]_{m_1}\ldots[b_n]_{m_n}}{\ldots[\alpha b_1 b_2\ldots b_n]_{M_n}\ [q]_{m_1}\ldots[q]_{m_n}} \times$$

$$\times (x/b_1b_2\ldots b_{n-1})^{m_1}(x/b_2b_3\ldots b_{n-1})^{m_2}\ldots$$

$$(x/b_{n-1})^{m_{n-1}}\, x^{m_n}$$

$$= \Phi\begin{bmatrix}(a), b_1b_2\ldots b_n,\ \alpha b_1;\ x/b_1b_2\ldots b_{n-1}\\(c),\ \alpha b_1b_2\ldots b_n\end{bmatrix}$$

and

$$\sum_{m_1,\ldots,m_n\geq 0} \frac{[(a)]_{M_1}[d_2\ldots d_{n-1}b_n]_{M_2}}{[(c)]_{M_1}[b_2d_3\ldots d_{n-1}b_n]_{M_2}} \times \frac{[d_3\ldots d_{n-1}b_n]_{M_3}\ldots}{[b_3d_4\ldots d_{n-1}b_n]_{M_3}\ldots} \times \frac{\ldots[d_{n-1}b_n]_{M_{n-1}}\;[b_1]_{m_1}\ldots[b_n]_{m_n}}{\ldots[b_{n-1}b_n]_{M_{n-1}}\;[q]_{m-1}\ldots[q]_{m_n}} \times (xd_2\ldots d_{n-1}b_n)^{m_1} \quad (2)$$

$$\times (xd_3\ldots d_{n-1}b_n)^{m_2}\ldots(xd_{n-1}b_n)^{m_{n-2}}(xb_n)^{m_{n-1}}x^{m_n}$$

$$= \phi\begin{bmatrix}(a),b_1d_2\ldots d_{n-1}b_n;x\\(c)\end{bmatrix},$$

where in (2.1) and (2.2) above $|x|<1$ and

$$M_r = m_r + m_{r+1} + \ldots + m_n \quad (r = 1,2,\ldots,n).$$

The proof of (2.1) follows by simple series manipulation and repeated use of the q-analogue of Saalschütz's theorem (cf. Slater [5;IV.4] p. 247). The proof of (2.2) also follows similarly when we make repeated use of the q-analogue of Gauss's theorem (cf. Slater [5; IV. 3], p.247).

&3. SUMMATIONS

In this section we shall discuss the deduction of certain general summations of q-series. It is clearly evident that if we can sum the ϕ-series on the right of (2.1) and (2.2) for particular values of the parameters, it leads to the summation of a q-series on their left side. The veriety of parameters $a_1,a_2,\ldots,a_A,c_1,c_2,\ldots,c_C,b_1,b_2,\ldots,b_n$ and the argument x give us a very long range of choice for summing the ϕ-series with the help of the known summations leading to numerous q-series summations of more complicated nature.

We deduce below some general cases to illustrate our assertion. Let us first replace x by $xb_1b_2\ldots b_{n-1}$ in(2.1) and then take A = 8 = C and put $a_1 = a$, $a_2 = q\sqrt{a}$, $a_3 = -q\sqrt{a}$, $a_4 = b$, $a_5 = c$, $a_6 = d$, $a_7 = e$, $a_8 = \alpha q^{-r_1-.-r_n}$, $c_1 = \sqrt{a}$, $c_2=-\sqrt{a}$, $c_3= cq/b$, $c_4 = aq/c$, $c_5 = aq/d$, $c_6 = aq/e$, $c_7=aq^{1+r_1+\ldots+r_n}$, $c_8 = \alpha q^{-r_1}$, $b_1 = q^{-r_1},\ldots, b_n= q^{-r_n}$ ($r_1,r_2,\ldots, r_n$ being non-negative integers), x = q and set $q^{1+R}a^2 =$ bcde ($R = r_1 +\ldots+ r_n$) and then sum the resulting well-poised ${}_8\phi_7$-series on the right with the help of Jackson's theorem (cf. Slater [5; IV.8], p.247 , we get

$$\sum_{m_1,\ldots m_n\geq 0} \frac{[a]_{M_1}[q\sqrt{a}]_{M_1}[-q\sqrt{q}]_{M_1}[b]_{M_1}[c]_{M_1}}{[\sqrt{a}]_{M_1}[-\sqrt{a}]_{M_1}[aq/b]_{M_1}[aq/b]_{M_1}[aq/c]_{M_1}} \times$$

$$\frac{[d]_{M_1}[e]_{M_1}[\alpha q^{-r_1-\ldots-r_n}]_{M_1}[\alpha]_{M_2}[\alpha q^{-r_1}]_{M_3}}{[aq/d]_{M_1}[aq/e]_{M_1}[aq^{1+r,+\ldots+r_n}]_{M_1}[\alpha q^{-r_1}]_{M_1}} \times$$

$$\times \frac{[\alpha q^{-r_1-r_2}]_{M_4}\ldots[\alpha q^{-r_1-\ldots r_{n-2}}]_{M_n}[q^{-r_1}]_{m_1}\ldots}{[\alpha q^{-r_1-r_2}]_{M_2}\ldots[\alpha q^{-r_1\ldots r_n}]_{M_n}[q]_{m_1}\ldots} \times$$

$$\times \frac{\ldots[q^{-r_n}]_{M_n}}{\ldots[q]_{m_n}}$$

$$\cdot \times q^{M_1-\{r_1m_2+(r_1+r_2)m_3+\ldots+(r_1+r_2+\ldots+r_{n-1})m_n\}}$$

$$= \frac{[aq]_R\,[aq/bc]_R\,[aq/cd]_R\,[aq/bd]_R}{[aq/b]_R\,[aq/c]_R\,[aq/d]_R\,[aq/bcd]_R},$$

where $M_s = m_s+m_{s+1}+\ldots+m_n$ ($s = 1,2,\ldots,n$).

Now, if we take $\alpha = o$ in the above summation (3.1) and let $q \to 1$, we get a known summation due to Srivastava [6;(3,3)].

Next, if we first replace x by $x\, b_1 b_2 \ldots b_n$ in (2.1) and then take $A=8=C$, $a_1=a$, $a_2=q\sqrt{a}$, $a_3=-q\sqrt{a}$, $a_4=b$, $a_5=c$, $a_6=d$, $a_7=q^{-N}$ (N being a non-negative integer), $a_8=\alpha b_1 b_2 \ldots b_n$, $c_1=\sqrt{a}$, $c_2=-\sqrt{a}$, $c_3=aq/b$, $c_4=aq/c$, $c_5=aq/d$, $c_6=aq^{1+N}$, $c_7=aq/b_1 b_2 \ldots b_n$, $c_8=\alpha b_1$ and $x=q$ and then sum resulting ${}_8\phi_7$ on the right with the help of Jackson's theorem (cf. Slater [5; IV.8], p.247), we get, for $q^{1+N} a^2 = bcd b_1 b_2 \ldots b_n$,

$$\sum_{m_1,\ldots,m_n \geq o} \frac{[a]_{M_1}[q\sqrt{a}]_{M_1}[-q\sqrt{a}]_{M_1}[b]_{M_1}[c]_{M_1}[d]_{M_1}}{[\sqrt{a}]_{M_1}[-\sqrt{a}]_{M_1}[aq/b]_{M_1}[aq/c]_{M_1}[aq/d]_{M_1}} \times \tag{2}$$

$$\times \frac{[q^{-N}]_{M_1}[\alpha b_1 \ldots b_n]_{M_1}[\alpha]_{M_2}[\alpha b_1]_{M_3}[\alpha b_1 b_2]_{M_4}}{[aq^{1+N}]_{M_1}[\alpha b_1]_{M_1}[\alpha b_1 b_2]_{M_2}[\alpha b_1 b_2 b_3]_{M_3} \ldots} \times$$

$$\times \frac{\ldots [\alpha b_1 b_2 \ldots b_{n-2}]_{M_n}[b_1]_{m_1} \ldots [b_n]_{m_n}}{\ldots [\alpha b_1 b_2 \ldots b_n]_{M_n}[q]_{m_1} \ldots [q]_{m_n}} \times$$

$$\times\, q^{M_1} b_1^{m_2} (b_1 b_2)^{m_3} \ldots (b_1 b_2 \ldots b_{n-1})^{m_n}$$

$$= \frac{[aq]_N [aq/bc]_N [aq/cd]_N [aq/bd]_N}{[aq/b]_N [aq/c]_N [aq/d]_N [aq/bcd]_N} .$$

The summations of the type (3.1) and (3.2) do not appear in the literature. The multiplicity of the series on the left of these summations depends on the value of n. If we take $\alpha=o$ in (3.2), we get a summation of the q-analogue of Lauricella's F_D-function.

One can also make use of (2.2) to deduce summations for q-series.

ACKNOWLEDGMENT

This work has been supported by the council of Scientific and Industrial Research, Government of India, New Delhi, under a research scheme.

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Recibido el 7 de enero de 1987