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FRACTIONAL CALCULUS AND IT'S APPLICATIONS TO THE NON-HOMOGENEOUS GAUSS' EQUATIONS

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ABSTRACT

Some other many papers on fractional calculus have been reported by the author. In this paper, firstly, we will show a table of fractional differintegrations of elementary functions which is obtained obeying the definition of author.

Next we will show an application of our fractional calculus to the non-Homogeneous Gauss' equation which is a differential equation of Fuchs type.

RESUMEN

El autor ha publicado varios trabajos sobre el cálculo fraccional. En este trabajo primero, se da una tabla de 'diferintegración' de algunas funciones elementales, usando la definición del autor. Además se resuelve la ecuación diferencial no-homogénea de Gauss mediante la aplicación del cálculo fraccional.

&0. INTRODUCTION

(Definition of fractional calculus)

Definition. If $f(z)$ is a regular function and it has no branch point inside C and on $C(C = \{\underline{C}, \underset{+}{C}\}$, $\underline{C}$ is an integral curve along the cut joining two points z and $-\infty + i\mathrm{Im}(z)$, and $\underset{+}{C}$ is an integral curve along the cut joining two points z and $\infty + i\mathrm{Im}(z)$),

$$f_\nu = {}_Cf_\nu(z) = \frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta-z)^{\nu+1}}\, d\zeta$$

Γ : Gamma function

$\nu \notin Z^-$, $\nu \in R$

and

$$f_{-n} = \lim_{\nu \to -n} f_\nu \qquad (n \in Z^+)$$

where $\zeta \neq z$, $-\pi \leq \arg(\zeta - z) \leq \pi$ for $\underline{C}$ and $0 \leq \arg(\zeta - z) \leq 2\pi$ for $\underset{+}{C}$, then $f_\nu(\nu>0)$ is the fractional derivative of order ν and $f_\nu(\nu<0)$ is the fractional integral of order $-\nu$, if f_ν exists (consider the principal value of f for many valued function)

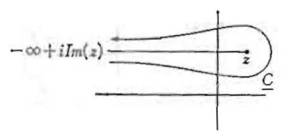

Fig. 1

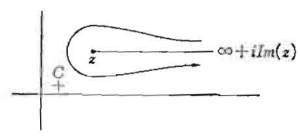

Fig. 2

&1. TABLE OF FRACTIONAL DIFERINTEGRATIONS OF ELEMENTARY FUNCTIONS AND SOME LEMMAS

Through the author's definition for fractional differintegration, we have Table 1. And, to make sure, we will show List 1. [1] [2] [9]

Lemma 1. Let $f(z) = f$ be regular and one valued functions. If $f_\nu(\neq 0)$ and $f_\mu(\neq 0)$ exist, then

$$(f_\nu)_\mu = (f_\mu)_\nu = f_{\mu+\nu} \quad \text{(index law)} \qquad (1)$$

Lemma 2. Let $u(z)$ and $v(z)$ be regular and one valued functions. If u_ν and v_ν exist, then

$$(u.v)_\nu = \sum_{n=0}^{\infty} \frac{\Gamma(\nu+1)}{\Gamma(\nu-n+1)\,\Gamma(n+1)} u_{\nu-n} v_n \,. \qquad (2)$$

Refer [6], [10] and [21] for these Lemmas

Table 1. Nishimoto's Fractional differintegrations of elementary functions

	$f(z)$	$f_\nu(z)$	
1.	1	0 $\quad (\nu \neq -m,\ m \in Z^+ \cup \{0\})$	
2.	e^{az} $(a \neq 0)$	$a^\nu e^{az}$	
3.	e^{-az} $(a \neq 0)$	$e^{-i\pi\nu} a^\nu e^{-az}$	
4.	$\cosh az$ $(a \neq 0)$	$(-ia)^\nu \cosh\left(az + i\frac{\pi}{2}\nu\right)$	
5.	$\sinh az$ $(a \neq 0)$	$(-ia)^\nu \sinh\left(az + i\frac{\pi}{2}\nu\right)$	
6.	$\cos az$ $(a \neq 0)$	$a^\nu \cos\left(az + \frac{\pi}{2}\nu\right)$	
7.	$\sin az$ $(a \neq 0)$	$a^\nu \sin\left(az + \frac{\pi}{2}\nu\right)$	
8.	z^a	$e^{-i\pi\nu} \frac{\Gamma(\nu-a)}{\Gamma(-a)} z^{a-\nu} \left(\left\| \frac{\Gamma(\nu-a)}{\Gamma(-a)} \right\| < \infty \right)$	
9.	$\log az$ $(a \neq 0)$	$-e^{-i\pi\nu}\Gamma(\nu) z^{-\nu} \quad (\|\Gamma(\nu)\| < \infty)$	
10.	$\log z$	$\frac{1}{m!} z^m \log z + z^m \sum_{k=0}^{m-1} \frac{(-1)^{m-k}}{k!\{(m-k)!\}(m-k)}$ $(\nu = -m,\ m \in Z^+)$	
11.	$(z^2-1)^\nu$ $(\arg\sqrt{z^2-1} = \phi)$	$\frac{2^\nu \Gamma(\nu+1)}{2\pi} \int_{-\pi}^{\pi} \{z + \sqrt{z^2-1} \cos(\theta-\phi)\}^\nu d\theta$	$Re\ \nu > 0$, $z \neq \pm 1$
		$2^\nu \Gamma(\nu+1) P_\nu(z) \qquad (\phi = 0)$	
		$\frac{2^\nu}{\Gamma(-\nu)} \int_{-\infty}^{\infty} \{-z + \sqrt{z^2-1} \cosh(\varphi - i\phi)\}^\nu d\varphi$	$Re\ \nu < 0$, $z \neq \pm 1$
		$\frac{2^\nu}{\Gamma(-\nu)} \cdot 2Q_\nu(-z) \qquad (\phi = 0,\ z(\text{real}) < -1)$	
12.	$z^{\nu-\lambda}(1-z)^{\lambda-\alpha}$	$e^{i\pi\{-\alpha\pm(\lambda-\nu)\}} \cdot \frac{\Gamma(\alpha)}{\Gamma(\alpha-\nu)} z^{-\alpha} {}_2F_1\left(\alpha,\ \alpha-\lambda:\ \alpha-\nu;\ \frac{1}{z}\right)$ $\left(+ \text{for } \|\arg z\| < \frac{\pi}{2},\ - \text{for } \frac{\pi}{2} < \|\arg z\| < \pi \text{ for double sign } \pm \text{ and } \|z\| > 1. \right)$	

(See the footnote of List 1.)

List 1.

Functions $f(x)$	1	x^a	e^{ax}	$\cos ax$	$\log x = \ln x$	Note
f_ν	1_ν	$(x^a)_\nu$	$(e^{ax})_\nu$	$(\cos ax)_\nu$	$(\log x)_\nu$	
fractional *integrals* fo Riemann & Liouville	$\frac{x^\nu}{\Gamma(\nu+1)}$	$\frac{\Gamma(a+1)}{\Gamma(a+1+\nu)} x^{\nu+a}$ $Re\ (a+1) > 0$				integrals for $Re\ \nu > 0$
fractional *integrals* of Weyl		$\frac{\Gamma(-a-\nu)}{\Gamma(-a)} x^{\nu+a}$ $0 < Re\ \nu < Re\ (-a)$	$(-a)^{-\nu} e^{ax}$ $Re(-ax) > 0$	$a^{-\nu} \cos\left(ax + \frac{\pi}{2}\nu\right)$ $a > 0,\ 0 < Re\ \nu < 1$		integrals for $Re\ \nu > 0$
fractional *derivatives* of Osler		$\frac{\Gamma(a+1) x^{a-\nu}}{\Gamma(a-\nu+1)}$	$\frac{{}_1F_1(1;\ 1-\nu;\ ax)}{\Gamma(1-\nu)} \times x^{-\nu}$	$\frac{x^{-\nu}}{2\Gamma(1-\nu)} \times [{}_1F_1(1;\ 1-\nu;\ iax) + {}_1F_1(1; 1-\nu; -iax)]$	$\frac{x^{-\nu}}{\Gamma(1-\nu)} \times [\log x - \gamma - \psi(1-\nu)]$	derivatives for $Re\ \nu > 0$
fractional *differintegrals* of Nishimoto	0 for $\nu \notin Z^- \cup \{0\}$	$e^{-i\pi\nu} \frac{\Gamma(\nu-a)}{\Gamma(-a)} x^{a-\nu}$ $\left\| \frac{\Gamma(\nu-a)}{\Gamma(-a)} \right\| < \infty$	$a^\nu e^{ax}$ $a \neq 0$	$a^\nu \cos\left(ax + \frac{\pi}{2}\nu\right)$ $a \neq 0$	$-e^{-i\pi\nu}\Gamma(\nu) x^{-\nu}$ for $\nu \notin Z^- \cup \{0\}$; $\frac{1}{m!} x^m \log x + x^m \times \sum_{k=0}^{m-1} \frac{(-1)^{m-k}}{k! \cdot (m-k)! \cdot (m-k)}$ for $\nu = -m,\ m \in Z^+$	derivatives for $Re\ \nu > 0$, integrals for $Re\ \nu < 0$.

Note 1. Z^+: set of the positive integers, Z^-: set of the negative integers.

Note 2. $(1)_\nu$ diverge for $Re\ \nu < 0$ by the direct calculation (obeying author's definition). However we calculate as $(1)_\nu = (z^0)_\nu$.

Note 3. In case of $a \in Z^+ \cup \{0\}$ and $\nu \in Z^-$, calculate as (for example $a = 2,\ \nu = -1$)

$$(z^2)_{-1} = \lim_{\nu \to -1} (z^2)_\nu = \lim_{\nu \to -1} e^{-i\pi\nu} \frac{\Gamma(\nu-2)}{\Gamma(-2)} z^{2-\nu} = e^{i\pi} \frac{\Gamma(-3)}{\Gamma(-2)} z^3 = \frac{\Gamma(-3)}{3\Gamma(-3)} z^3 = \frac{1}{3} z^3 .$$

&2. SOLUTIONS TO (HOMOGENEOUS AND NON-HOMOGENEOUS) HYPERGEOMETRIC EQUATIONS OF GAUSS

(I) Theorem 1. If $f_{[illegible]}$ ($\neq 0$) exists, then the non-homogeneous hypergeometric differential equation

$$[illegible].(z^2-z) + [illegible] + \phi.\alpha[illegible] = f \qquad ([illegible] \neq 0,1) \qquad (1)$$

has a partic[illegible] the form

$$[illegible] = ((f_{-[illegible]}.z^{[illegible]} [illegible] (z-1)^{\gamma-\beta-1})_{[illegible]-1}, \qquad (2)$$

where [illegible], $f = f(z)$, $z \in$ [illegible], and α, β and γ are constants [17].

Proof. Putting

$$[illegible] = w_{[illegible]}, \qquad (3)$$

yields

$$[illegible] = w_{[illegible]+1} \text{ (by Lemma 1.)} \qquad (4)$$

and

$$[illegible] = w_{[illegible]+[illegible]}, \qquad (5)$$

where $w = w(z)$

Substituting (3), (4) and (5) into (1). we obtain

$$w_{\alpha+2}.z^2 - w_{\alpha+2}.z + w_{\alpha+1}.z(\alpha+\beta+1) - w_{\alpha+1}.\gamma + w_{\alpha}.\alpha\beta = f, \qquad (6)$$

that is

$$(w_2.z^2)_{\alpha} - (w_2.z)_{\alpha} + (w_1.z)_{\alpha}.(1+\beta-\alpha) + (w_1)_{\alpha}.$$
$$.(\alpha-\gamma) = f. \qquad (7)$$

Consequently we have

$$w_2.(z^2-z) + w_1.\{z(1+\beta-\alpha) + (\alpha-\gamma)\} = f_{-\alpha} \qquad (8)$$

from (7), since

$$(w_m.z^n)_{\alpha} = \sum_{k=o}^{n} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-k)\,\Gamma(k+1)} (w_m)_{\alpha-k}.(z^n)_k, \qquad (9)$$

where $n \in Z^+ \cup \{0\}$, (by Lemma 2.)

Put

$$w_1 = u \qquad (10)$$

in (8), we have then

$$u_1 + u.\frac{z(1+\beta-\alpha) + (\alpha-\gamma)}{z^2-z} = f_{-\alpha}.\frac{1}{z^2-z}. \qquad (11)$$

A particular solution of this linear differential equation of first order is given by

$$u = (f_{-\alpha}.z^{\gamma-\alpha-1}(z-1)^{\beta-\gamma})_{-1}.z^{\alpha-\gamma}(z-1)^{\gamma-\beta-1}. \qquad (12)$$

Therefore we obtain, using (10) and (3),

$$\phi = w_{\alpha} = (u_{-1})_{\alpha} = u_{\alpha-1} \qquad (13)$$

$$= ((f_{-\alpha}.z^{\gamma-\alpha-1}(z-1)^{\beta-\gamma})_{-1}.z^{\alpha-\gamma}(z-1)^{\gamma-\beta-1})_{\alpha-1} \qquad (14)$$

as a particular solution to the equation (1).

Inversely we have

$$\phi_1 = u_{\alpha} \qquad (15)$$

and

$$\phi_2 = u_{\alpha+1} \qquad (16)$$

from (13). Substituting (13), (15) and (16) into the left hand side of (1), we obtain

$$\text{L.H.S. of (1)} = (u_1.z^2)_{\alpha} - (u_1.z)_{\alpha} + (u.z)_{\alpha}.(1+\beta-\alpha) +$$
$$+ (u)_{\alpha}.(\alpha-\gamma) \qquad (17)$$

$$= (u_1.(z^2-z) + u. \{z(1+\beta-\alpha) + (\alpha-\gamma)\})_\alpha \quad (18)$$

$$= (f_{-\alpha})_\alpha \quad \text{(see (8))} \quad (19)$$

$$= f. \quad (20)$$

Changing the order

$$(f_{-\alpha}.z^{\gamma-\alpha-1}(z-1)^{\beta-\gamma})_{-1} \quad \text{and} \quad z^{\alpha-\gamma}(z-1)^{\gamma-\beta-1}$$

in (2), we have other solution

$$\phi = (z^{\alpha-\gamma}(z-1)^{\gamma-\beta-1}.(f_{-\alpha}.z^{\gamma-\alpha-1}(z-1)^{\beta-\gamma})_{-1})_{\alpha-1} \quad (2)'$$

for $\alpha \notin Z$.

Put α and β instead of β and α respectively in (2) and in (2)', we have then following two other solutions for $\beta \notin Z$, if $\alpha \neq \beta$.

$$\phi = ((f_{-\beta}.z^{\gamma-\beta-1}(z-1)^{\alpha-\gamma})_{-1}.z^{\beta-\gamma}(z-1)^{\gamma-\alpha-1})_{\beta-1}, \quad (21)$$

$$\phi = (z^{\beta-\gamma}(z-1)^{\gamma-\alpha-1}.(f_{-\beta}.z^{\gamma-\beta-1}(z-1)^{\alpha-\gamma})_{-1})_{\beta-1} \quad (21)$$

(II) Theorem 2.

$$\phi_2.(z^2-z) + \phi_1. \{z(\alpha+\beta+1) - \gamma\} + \phi.\alpha\beta = 0$$

$$(z \neq 0,1) \quad (22)$$

has a solution of the form

$$\phi = (z^{\alpha-\gamma}(z-1)^{\gamma-\beta-1})_{\alpha-1} , \quad (23)$$

where $\phi = \phi(z)$ and $z \in C$. The equation (22) is hypergeometric differential equation of Gauss [17].

Proof. Putting

$$\phi = w_\alpha \quad (24)$$

in (22), we have then (see the proof in (I))

$$w_2.(z^2-z) + w_1. \{z(1+\beta-\alpha) + (\alpha-\gamma)\} = 0 \quad (25)$$

A particular solution of this equation is given as follows.

$$w = (z^{\alpha-\gamma}.(z-1)^{\gamma-\beta-1})_{-1}. \quad (26)$$

Substituting (26) into (24), we have then

$$\phi = w_\alpha = (z^{\alpha-\gamma}.(z-1)^{\gamma-\beta-1})_{\alpha-1} \quad (27)$$

Inversely we obtain

$$\phi_1 = w_{\alpha+1} \quad (28)$$

and

$$\phi_2 = w_{\alpha+2} \quad (29)$$

Substituting (27), (28) and (29) into the left hand side of (22), we have then

$$\text{L.H.S. of (22)} = ((w_2.z^2-w_2.z+w_1.z(1+\beta-\alpha) + w_1.(\alpha-\gamma))_\alpha \quad (30)$$

$$= (w_2.(z^2-z) + w_1 . \{z(1+\beta-\alpha) + (\alpha-\gamma)\})_\alpha \quad (31)$$

$$= (z^{\alpha-\gamma}(z-1)^{\gamma-\beta-1} \{(\alpha-\gamma)(z-1)+(\gamma-\beta-1)z + (1+\beta-\alpha)z + (\alpha-\gamma)\})_\alpha \quad (32)$$

$$= (0)_\alpha \quad (33)$$

$$= 0 . \quad (34)$$

Changing α and β in (27), we have other solution

$$\phi = (z^{\beta-\gamma}(z-1)^{\gamma-\alpha-1})_{\beta-1}, \qquad (35)$$

if $\alpha \neq \beta$.

(III) Theorem 3. If $f_\alpha (\neq 0)$ exists, then the fractional differintegrated function

$$\phi = ((f_{-\alpha}.z^{\gamma-\alpha-1}(z-1)^{\beta-\gamma})_{-1}.z^{\alpha-\gamma}(z-1)^{\gamma-\beta-1})_{\alpha-1} +$$

$$+ (z^{\alpha-\gamma}.(z-1)^{\gamma-\beta-1})_{\alpha-1} \qquad (36)$$

satisfies the differential equation of Fuchs type (1), where $z \in C$.

Proof. It is clear by the Theorems 1 and 2.

&3. TRUE COLORS OF GAUSS' HYPERGEOMETRIC FUNCTIONS

Theorem 4. We have

(i) $F(\beta-\gamma+1, \beta; \beta-\alpha+1; 1/z)$

$$= -e^{i\pi\alpha} \frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)} z^{\beta}(z^{\alpha-\gamma}(z-1)^{\gamma-\beta-1})_{\alpha-1} \qquad (1)$$

for $|z| > 1$, and

(ii) $F(\alpha-\gamma+1, \beta-\gamma+1; 2-\gamma; z)$

$$= e^{i\pi(\alpha+\beta-\gamma)} \frac{\Gamma(\gamma-\alpha)}{\Gamma(\gamma-1)} z^{\gamma-1}(z^{\alpha-\gamma}(z-1)^{\gamma-\beta-1})_{\alpha-1} \qquad (2)$$

for $|z| < 1$.

Proof of (i). We have

$$(z-1)^{\lambda} = z^{\lambda}(1-\frac{1}{z})^{\lambda}$$

$$= z^{\lambda} \sum_{k=o}^{\infty} \frac{(-1)^k \Gamma(\lambda+1)}{\Gamma(k+1)\Gamma(\lambda-k+1)} z^{-k} \quad (\text{for } |z|>1) \qquad (3)$$

hence we obtain

$$(z^{\alpha-\gamma}(z-1)^{\gamma-\beta-1})_{\alpha-1}$$

$$= (z^{\alpha-\beta-1} \sum_{k=o}^{\infty} \frac{(-1)^k \Gamma(\gamma-\beta)}{\Gamma(k+1)\Gamma(\gamma-\beta-k)} z^{-k})_{\alpha-1} \qquad (4)$$

$$= \sum_{k=o}^{\infty} \frac{(-1)^k \Gamma(\gamma-\beta)}{\Gamma(k+1)\Gamma(\gamma-\beta-k)} (z^{\alpha-\beta-1-k})_{\alpha-1} \qquad (5)$$

$$= -e^{-i\pi\alpha} z^{-\beta} \sum_{k=o}^{\infty} \frac{(-1)^k \Gamma(\gamma-\beta)\Gamma(\beta+k)}{\Gamma(k+1)\Gamma(\gamma-\beta-k)\Gamma(\beta+1+k-\alpha)} z^{-k} \qquad (6)$$

$$= -e^{-i\pi\alpha} \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)} z^{-\beta} F(\beta-\gamma+1, \beta; \beta-\alpha+1; \frac{1}{z}) \qquad (7)$$

for $|z| > 1$. Therefore we have (i) from above result (7).

Proof of (ii). We have

$$(z-1)^{\lambda} = e^{i\pi\lambda}(1-z)^{\lambda}$$

$$= e^{i\pi\lambda} \sum_{k=o}^{\infty} \frac{(-1)^k \Gamma(\lambda+1)}{\Gamma(k+1)\Gamma(\lambda+1-k)} z^k \quad (\text{for } |z| < 1), \qquad (8)$$

hence we obtain

$$(z^{\alpha-\gamma}(z-1)^{\gamma-\beta-1})_{\alpha-1}$$

$$= e^{i\pi(\gamma-\beta-1)} \sum_{k=o}^{\infty} \frac{(-1)^k \Gamma(\gamma-\beta)}{\Gamma(k+1)\Gamma(\gamma-\beta-k)} (z^{k+\alpha-\gamma})_{\alpha-1} \qquad (9)$$

$$= e^{i\pi(\gamma-\alpha-\beta)} z^{1-\gamma} \sum_{k=o}^{\infty} \frac{(-1)^k \Gamma(\gamma-\beta) \ \Gamma(\gamma-k-1)}{(k+1) \ \Gamma(\gamma-\beta-k) \ \Gamma(\gamma-\alpha-k)} z^k \tag{10}$$

$$= e^{i\pi(\gamma-\alpha-\beta)} \frac{\Gamma(\gamma-1)}{\Gamma(\gamma-\alpha)} z^{1-\gamma} F(\alpha-\beta+1, \ \beta-\gamma+1; \ 2-\gamma; \ z) \tag{11}$$

for $|z|<1$. We have (ii) from (11).

&4. SOME EXAMPLES

Putting $\alpha=\nu$, $\beta=\nu-1$, and $\gamma=\nu+1$, we have

$$\phi_2.(z^2-z) + \phi_1.(2\nu z-\nu-1) + \phi.\nu(\nu-1) = f \quad (z \neq 0,1) \tag{1}$$

and

$$\phi = ((f_{-\nu} \cdot \frac{1}{(z-1)^2}). \frac{z-1}{z})_{\nu-1} \tag{2}$$

from the Theorem 1, and

$$\phi_2.(z^2-z) + \phi_1 .(2\nu z-\nu-1) + \phi.\nu(\nu-1) = 0 \quad (z \neq 0,1) \tag{3}$$

and

$$\phi = (z - \log z)_\nu \tag{4}$$

from the Theorem 2, respectively [15].

(I) More practically, let $\nu = -1/2$ and $f = z^{-1/2}$, we have then

$$\phi_2.(z^2-z) - \phi_1.(z+\frac{1}{2}) + \phi.\frac{3}{4} = z^{-1/2} \quad (z\neq 0,1) \tag{5}$$

from (1), hence a particular solution to this differential equation is given as

$$\phi = \left(\left((z^{-1/2})_{1/2} \cdot \frac{1}{(z-1)^2}\right)_{-1} \cdot \frac{z-1}{z}\right)_{-3/2} \tag{6}$$

$$= \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} G(k).z^{(1/2)-k} \quad (\text{for } |z|>1) \tag{7}$$

by (2), where

$$G(K) = \Gamma(k-(1/2))/k \ \Gamma(k+2), \tag{8}$$

since

$$(z^a)_\nu = e^{-i\pi\nu} \frac{\Gamma(\nu-a)}{\Gamma(-a)} z^{a-\nu} \left(\left| \frac{\Gamma(\nu-a)}{\Gamma(-a)} \right| < \infty \right) \tag{9}$$

and

$$\frac{1}{z(z-1)^2} = \sum_{k=1}^{\infty} kz^{-(k+2)} \quad (|z| > 1). \tag{10}$$

Inversely,

(i) Putting

$$w = \left((f_{1/2} \cdot \frac{1}{(z-1)^2})_{-1} \cdot \frac{z-1}{z}\right)_{-1}, \quad f = z^{-1/2} \tag{11}$$

in (6), we have

$$\phi = w_{-1/2} \tag{12}$$

hence

$$\phi_1 = w_{1/2} \tag{13}$$

and

$$\phi_2 = w_{3/2}. \tag{14}$$

Substituting (12), (13) and (14) into the left hand side of (5), we have then

$$\text{L.H.S. of (5)} = w_{3/2}\cdot(z^2-z) - w_{1/2}\cdot(z+\frac{1}{2}) +$$

$$+ w_{-1/2}\cdot\frac{3}{4} \qquad (15)$$

$$= (w_2 . z^2)_{-1/2} - (w_1 . z)_{1/2} \qquad (16)$$

$$= (w_2 . z^2)_{-1/2} - ((w_1 . z)_1)_{-1/2} \qquad (17)$$

$$= (f_{1/2})_{-1/2} \qquad (18)$$

$$= z^{-1/2} \qquad (19)$$

(ii) Next, we have

$$\phi_1 = \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} G(k)\ (\frac{1}{2} - k) z^{-k-(1/2)} \qquad (20)$$

and

$$\phi_2 = \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} G(k)(\frac{1}{2} - k)(-\frac{1}{2} - k) z^{-k-(3/2)} \qquad (21)$$

from (7).

Substituting (7), (20) and (21) into the left hand side of (5), we have then

$$\text{L.H.S. of (5)} = \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} G(k)(k^2-k) z^{(1/2)-k}$$

$$- \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} G(k)(k^2 - \frac{1}{2} k) z^{-(1/2)-k} \qquad (22)$$

$$= \frac{2}{\sqrt{\pi}} G(1) z^{-1/2} + \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} \left[G(k+1)\ \{(k+1)^2 + (k+1)\}\right.$$

$$\left. - G(k)(k^2 - \frac{1}{2} k)\right]\ z^{-(1/2)-k} \qquad (23)$$

$$= z^{-1/2}, \qquad (24)$$

since

$$G(1) = \sqrt{\pi}/2,$$

and

$$G(k+1)\ \{(k+1)^2 + (k+1)\} = G(k)\ (k^2 - \frac{1}{2} k)$$

$$= \Gamma(k+(1/2))/\ \Gamma(k+2)\ , \qquad (25)$$

again

(II) (i) Let $\nu = -1/2$ in (3), we have then

$$\phi_2 . (z^2-z) - \phi_1 . (z + \frac{1}{2}) + \phi . \frac{3}{4} = 0 \qquad (26)$$

A particular solution to this Gauss' equation is given as

$$\phi = (z - \log z)_{-1/2} \qquad (27)$$

$$= i\ \Gamma(-1/2) z^{1/2} \ \text{(see Table 1) [7] [10]} \qquad (28)$$

by (4). Inversely we have

$$\phi_1 = i\ \Gamma(-1/2) z^{-1/2}\ /\ 2 \qquad (29)$$

and

$$\phi_2 = -i\ \Gamma(-1/2) z^{-3/2}\ /4 \qquad (30)$$

from (28).

Substituting (28), (29) and (30) into the left hand side of (26), we obtain

$$\text{L.H.S. of (26)} = i\Gamma(-1/2)\left(-\frac{1}{4}z^{1/2}+\frac{1}{4}z^{-1/2}\right.$$

$$\left.-\frac{1}{2}z^{1/2}-\frac{1}{4}z^{-1/2}+\frac{3}{4}z^{1/2}\right) = 0. \quad (31)$$

(ii) Let $\nu = 2$ in (3) and (4), we have then

$$\phi_2.(z^2-z) + \phi_1.(4z-3) + \phi.2 = 0 \qquad (z\neq 0,1) \quad (32)$$

and

$$\phi = (z - \log z)_2 = z^{-2} \quad (33)$$

respectively.

By assuming $\phi = \phi(z)z^{-2}$, the second solution independent to the first particular solution (33) is given by

$$\phi = -z^{-1} - z^{-2}\log(1-z) \qquad (z\neq 0,1). \quad (34)$$

On the other hand, assuming a solution of the form of power series, we have then

$$\phi = F(\nu,\nu-1 : \nu+1 ; z)$$

$$= \frac{\Gamma(\nu+1)}{\Gamma(\nu)\,\Gamma(\nu-1)}\sum_{n=o}^{\infty}\frac{\Gamma(\nu+n)\,\Gamma(\nu-1+n)}{\Gamma(\nu+1+n)\,\Gamma(n+1)}z^n \quad (|z|<1) \quad (35)$$

as a particular solution to the equation (3), where $F(\alpha,\beta;\gamma;z)$ is the Hypergeometric function.

Put $\nu = 2$ in (35), we have

$$\phi = 2\sum_{n=o}^{\infty}\frac{1}{2+n}z^n \qquad (|z|<1) \quad (36)$$

as a solution to the equation (32).

Solution (34) coincides with (36) except the coefficient of constant, since

$$\log(1-z) = -\sum_{n=1}^{\infty}\frac{1}{n}z^n \qquad (|z|<1). \quad (37)$$

However, the solution (34), in its form, is a more desirable and wide one than (36).

(iii) Let $\nu = -n(n\in Z^+ \cup \{0\})$, we have then

$$\phi_2.(z^2-z) + \phi_1.(-2nz+n-1) + \phi.n(n+1) = 0$$

$$(z\neq 0,1) \quad (38)$$

and

$$\phi = (z - \log z)_{-n} \quad (39)$$

from (3) and (4) respectively.

If $n = 0$, (38) and (39) are reduced to

$$\phi_2(z^2-z) - \phi_1 = 0 \qquad (z\neq 0,1) \quad (40)$$

and to

$$\phi = z - \log z \quad (41)$$

respectively. Clearly, (41) satisfies equation (40).

And in case of $n = 1$, for example, (38) and (39) are reduced to

$$\phi_2.(z^2-z) - \phi_1 \,.\, 2z + \phi.2 = 0 \qquad (z\neq 0,1) \quad (42)$$

and to

$$\phi = (z - \log z)_{-1} \quad (43)$$

respectively. Hence we have

$$\phi_1 = z - \log z \quad (44)$$

and

$$\phi_2 = 1 - z^{-1} \quad (45)$$

from (43). Substituting (43), (44) and (45) into the left hand side of (42), we have then

L.H.S. of (42) = 1 ≠ 0. (46)

That is, the function (43) does not satisfy the differential equation (42).

However, if we adopt

$$\phi = (z - \log z)_{-1}$$

$$= \int (z - \log z) dz$$

$$= \frac{1}{2} z^2 - (z \log z - z) + c \quad (47)$$

as a solution (c is an arbitrary constant of integration) to the equation (42) and substitute this into the L.H.S. of (42), we obtain

L.H.S. of (42) = 1 + 2c. (48)

Consequently, determining c = -1/2 in (47), we obtain

$$\phi = \frac{1}{2} z^2 - (z \log z - z) - \frac{1}{2} \quad (49)$$

as a solution to the equation (42). And (49) satisfies (42) clearly.

&.5. COMMENTS

In this paper, an application of "Fractional Calculus" to the solution of Gauss' Hypergeometric differential equation, which is a linear ordinary differential equation of Fuchs type, is shown as we see above.

By the method described here we can obtain very nice, brief and simple (in it's form) particular solutions, in which the differintegrated (by arbitrary order ν) function f_ν —— where f is the function lying on the right hand side of differential equation —— is contained explicitly, to non-homogeneous linear ordinary differential equations of Fuchs type. And the brief solutions give us very good prospects.

Of course, we can obtain nice, brief and simple solution (in its form) to the homogeneous linear ordinary differential equations of Fuchs type, again.

That is, the method reported here is a highly effective and a very powerful one used specially to obtain a particular solution to non-homogeneous linear ordinary differential equations of Fuchs type.

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