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ON MOMENTS OF PROBABILITY DISTRIBUTION FUNCTIONS AND H-FUNCTION TRANSFORM

ABSTRACT

We express the absolute moments of a probability distribution function in terms of the H-function transform. A special case and an example is cited to illustrate the application of our general result.

RESUMEN

En este trabajo se obtienen momentos absolutos de una función de distribución en términos de la función H. Se considera un caso especial y un ejemplo para ilustrar la aplicación de la fórmula general.

1. INTRODUCTION

If $f(x)$ is a probability distribution function and $\phi(p)$ is the Laplace-Stieltjes transform defined by

$$(1.1) \qquad \phi(p) = \int_0^\infty e^{-xp}\, df(x) \quad , \quad p\geq 0 \quad ,$$

then $\phi(p)$ possesses derivatives of all orders given by

$$(1.2) \qquad \int_0^\infty e^{-xp} x^n\, df(x) = (-1)^n \phi^n(p) \ , \ 0\leq p<\infty \ ,$$

where $f(x) = 0$ for $x<0$.

It follows in particular (see Feller [2]) that the probability distribution function $f(x)$ has a finite absolute moment of the $n^{\underline{th}}$ order iff $\phi^n(0)$ exists such that

$$(1.3) \qquad \int_0^\infty |x|^n\, d\, f(x) = (-1)^{-n} \phi^n(0).$$

Denote by A the class of all such functions f which are differentiable everywhere any number of times and if it and all of its derivatives are $0(x^{-\nu})$, for all ν as x increases without limit.

The Weyl fractional derivatives of a function $g(x)$ is defined as follows :

$$(1.4) \qquad {}_xD_\infty^\gamma\, g(x) = \frac{(-1)^\gamma}{\Gamma(-\gamma)} \int_x^\infty (t-x)^{-\gamma-1} g(t)\, dt \quad ,$$

for $\gamma<0$

For $\gamma \geq 0$,

$$(1.5) \qquad {}_xD_\infty^\gamma\, g(x) = \frac{d^r}{dx^r} \left({}_xD_\infty^{\gamma-r} g(x)\right) \ ,$$

where r is a positive integer such that $r-\gamma>0$. Whenever $g \in A$, the representations (1.4) and (1.5) exist (see Miller [5, p. 80]).

In a recent paper Wolfe [11] extended the law of moments of order n (n a positive integer) to an equivalent result valid for an arbitrary order λ (λ real). We find it convenient to record this result in the following form :

THEOREM 1 (Wolfe [11]). Let $f(x)$ be a probability distribution function such that $f(x)=0$ for $x<0$. Then $f(x)$ possesses an absolute moment of the λ^{th} order iff ${}_pD_\infty^\lambda\, g(0)$ exists such that

$$(1.6) \qquad \int_0^\infty x^\lambda\, d\, f(x) = (-1)^{-\lambda}\, {}_pD_\infty^\lambda\, g(0) \ , \ \lambda \ \underline{\text{real}} \ ,$$

where $g(p)$ denotes the Laplace-Stieltjes transform as defined in (1.1), of $f(x)$.

REMARK 1. It may be observed that the proof given by Wolfe [11, p.310] can also be formulated with the aid of an interesting relation giving the Laplace transform of the function $t^\gamma f(t)$, for all real γ , via Weyl fractional calculus, given in the paper of Raina and Koul [7, p.189].

Our object in this paper is to express the absolute moments of a probability distribution function in terms of the H-function transform, defined below, which provides a generalization of the aforecited Theorem 1. An example is cited to illustrate the application of our generalized result.

2. THE MAIN RESULT

If $f(x)$ is the probability distribution function such that $f(x) = 0$, for $x < 0$, then the integral equation

$$(2.1) \quad g(p) = \int_0^\infty H_{P,Q}^{M,N}\left[a(pt)^h \left| \begin{matrix}(a_i,\alpha_i)_{1,P}\\(b_i,\beta_i)_{1,Q}\end{matrix}\right.\right] d\,f(t)\,dt, \quad h>0,\ 0\le p<\infty,$$

defines an H-function transform of $f(t)$.

In particular, if

$$(2.2) \quad f(t) = \int_0^t \phi(x)dx, \quad 0<t<\infty,$$

then (2.1) reduces to the equivalent from of the H-function transform due to Gupta and Mittal [4, p. 142].

The H-function Kernel in (2.1) is the well known H-function of C.Fox [1, p.408] which we define in the following form :

$$(2.3) \quad H_{P,Q}^{M,N}\left[z\left| \begin{matrix}(a_i,\alpha_i)_{1,P}\\(b_i,\beta_i)_{1,Q}\end{matrix}\right.\right] = \frac{1}{2\pi w}\int_L \theta(s)\, z^s\, ds,$$

with $w = \sqrt{-1}$, and $\theta(s)$ given by

$$(2.4) \quad \theta(s) = \left(\prod_{j=1}^{M}\Gamma(b_j-\beta_j s)\prod_{j=1}^{N}\Gamma(1-a_j+\alpha_j s)\right)\left(\prod_{j=M+1}^{Q}\Gamma(1-b_j+\beta_j s)\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j s)\right)^{-1}.$$

The contour L is a suitably chosen contour of Mellin-Barnes type; the non negative integers M, N, P, Q satisfy the inequalities $0\le M\le Q$, $0\le N\le P$. The parameters α_i,β_i are all positive and the parameters a_i,b_i are arbitrary complex numbers.

It being understood that $(a_i,\alpha_i)_{1,P}$ condenses the array of parameters $(a_1,\alpha_1), \dots, (a_p,\alpha_p), p\ge 0$; with similar interpretation for $(b_i,\beta_i)_{1,Q}$. For a detailed account of the conditions of existence of the H-function (2.3), its various special cases and other important properties, we refer to the paper of Gupta and Jain [3].

Before stating our main result of this paper which provides a generalization of (1.6), we require the following interesting result :

LEMMA. Let $f(x)$ be a probability distribution function whose H-function transform $g(p)$ given by(2.1.) be such that $g(p) \in A$. Suppose

(i) $0\le M\le Q$, $0\le N\le P$ (M,N,P,Q being non-negative integers),

(ii) $h\left[(a_i-1)/\alpha_i\right] < 0$ $(i=1,\dots,N)$, $h>0$, $0\le p<\infty$,

(iii) $|\arg(a)| < \frac{1}{2}\Delta\pi$, $\Delta = \sum_1^M \beta_i - \sum_{M+1}^{Q}\beta_i + \sum_1^N\alpha_i - \sum_{N+1}^{P}\alpha_i > 0$.

Then $g(p)$ possesses derivarives of arbitrary order γ (real) given by

$$(2.5) \quad a\,(-1)^{-\gamma}\ {}_pD_\infty^\gamma\, g(p) = \Psi\left[x^\gamma f(x)\,;\,p\right],$$

where

$$(2.6) \quad \Psi\left[x^\gamma f(x);p\right] = \int_0^\infty H_{P+1,Q+1}^{M+1,N}\left[a(px)^h\left| \begin{matrix}(a_i-\gamma\frac{\alpha i}{h},\alpha_i)_{1,P},(-\gamma,h)\\(0,h),\ (b_i-\frac{\gamma\beta i}{h},\beta_i)_{1,Q}\end{matrix}\right.\right] x^\gamma df(x).$$

Proof : If $f(x)$ is a probability distribution function such that $f(x) = 0$ for $x<0$, then for $\gamma<0$, we have in view of (1.4) and (2.1),

$${}_pD_\infty^\gamma\, g(p) = \frac{(-1)^\gamma}{\Gamma(-\gamma)}\int_p^\infty (t-p)^{-\gamma-1}\, g(t)\,dt$$

$$= \frac{(-1)^\gamma}{\Gamma(-\gamma)}\int_p^\infty (t-p)^{-\gamma-1}\left(\int_0^\infty H_{P,Q}^{M,N}\left[a(tx)^h\left| \begin{matrix}(a_i,\alpha_i)_{1,P}\\(b_i,\beta_i)_{1,Q}\end{matrix}\right.\right] df(x)\right)dt$$

$$= \frac{(-1)^\gamma}{\Gamma(-\gamma)}\int_0^\infty\left(\int_0^\infty (t-p)^{-\gamma-1} H_{P,Q}^{M,N}\left[a(tx)^h\left| \begin{matrix}(a_i,\alpha_i)_{1,P}\\(b_i,\beta_i)_{1,Q}\end{matrix}\right.\right] dt\right) df(x).$$

Since

$$(2.7) \quad \int_p^\infty (x-p)^{-\gamma-1}\, x^\mu\, H_{P,Q}^{M,N}\left[ax^h\left| \begin{matrix}(a_i,\alpha_i)_{1,P}\\(b_i,\beta_i)_{1,Q}\end{matrix}\right.\right] dx$$

$$= p^{\mu-\gamma}\,\Gamma(-\gamma)\, H_{P+1,Q+1}^{M+1,N}\left[ap^h\left| \begin{matrix}(a_i,\alpha_i)_{1,P},(-\mu,h)\\(\gamma-\mu,h),(b_i,\beta_i)_{1,Q}\end{matrix}\right.\right]$$

provided that $h>0$, $Re\left[h(a_i-1)/\alpha_i\right] < Re(\gamma)<0$ $(i=1,\dots,N)$, $|\arg(a)|<\frac{1}{2}\Delta\pi$, Δ being given in condition (iii) (stated with Theorem 2 above).(which can rather easily be established by invoking the definitions (1.4) and (2.3) ; also deducible from the result given by Raina [9, p.40, Eq.(3.3)]}, we find that

$$(2.8)\quad {}_pD_\infty^\gamma\, g(p) = (-1)^\gamma \int_0^\infty p^{-\gamma}\, H^{M+1,N}_{P+1,Q+1}\left[a(px)^h \middle| \begin{matrix}(a_i,\alpha_i)_{1,P},(0,h)\\ (\gamma,h),(b_i,\beta_i)_{1,Q}\end{matrix}\right] df(x)$$

$$= \frac{(-1)^\gamma}{a} \int_0^\infty x^\gamma\, H^{M+1,N}_{P+1,Q+1}\left[a(px)^h \middle| \begin{matrix}(a_i-q\alpha_i/h,\alpha_i)_{1,P}\ ,\ (-\gamma,h)\\ (0,h)\ ,\ (b_i-q\beta_i/h,\beta_i)_{1,Q}\end{matrix}\right] x\ df(x),$$

which evidently establishes our result for $\gamma<0$.

For $\gamma\geq 0$, by invoking the definition (1.5) and the case (2.8) , we get

$$(2.9)\quad {}_pD_\infty^\gamma\, g(p) = \frac{(-1)^{\gamma-n}}{a}\frac{d^n}{dp^n}\left(\int_0^\infty x^{\gamma'} H^{M+1,N}_{P+1,Q+1}\left[a(px)^h \middle| \begin{matrix}(a_i-q'\frac{\alpha_i}{h},\alpha_i)_{1,P}\ ,(-q',h)\\ (\ 0\ ,\ h)\ ,\ (b_i-q'\frac{\beta_i}{h},\beta_i)_{1,Q}\end{matrix}\right] df(x),\right.$$

where $\gamma'=\gamma-n$

Differentiating under the sign of integral (justifiable under the conditions stated witn the theorem and the Lebesgue dominated convergence theorem), appealing to the formula of Skibiniski [10, p. 131, Eq.(4.1)] regarding the derivatives of H-function (see also Raina and Koul [6]), we arrive at the required result (2.5) for $\gamma\geq 0$. This proves our lemma.

REMARK 2. It must be mentioned that the aforecited lemma would also follow from a very recently established result due to Raina and Koul [8] , with of course, suitable amendments in the hypothesis and derivation method.

THEOREM 2. If f(x) is a probability distribution function such that $f(x) = 0$ for $x<0$ whose H-function transform g(p) given by (2.1) exist. Then g(p) possesses an absolute moment of the γ^{th} order $(-\infty<\gamma<\infty)$, iff ${}_pD_\infty^\gamma$ exists and is given by

$$(2.10)\quad \int_0^\infty x^\gamma\, df(x) = a(-1)^{-\gamma}\, \eta_P^Q(b_i,a_i;\gamma)\, {}_pD_\infty^\gamma g(0),$$

where

$$(2.11)\quad \eta_P^Q(b_i,a_i;\gamma) = \frac{\prod_{i=1}^{Q}\Gamma(1-b_i+\gamma)}{\Gamma(1+\gamma)\prod_{i=1}^{P}\Gamma(1-a_i+\gamma)}\ ,$$

$$|\arg(a)| < \frac{\pi(P-Q+1)}{2}\ ,$$

a_i , b_i are arbitrary complex parameters and assume values such that the Gamma quotients in (2.11) exist

Proof : suppose f(x) is a probability distribution function and let f(x) possess the absolute moment of the γ^{th} order $(-\infty<\gamma<\infty)$.

From the elementary relationship of the H-function [3, p. 600 Eq. (4.6)]

$$(2.12)\quad H^{1,P}_{P,Q+1}\left[z \middle| \begin{matrix}(a_i,1)_{1,P}\\ (b_i,1)_{1,Q}\end{matrix}\right] = \frac{\prod_{i=1}^{P}\Gamma(a_i)}{\prod_{i=1}^{Q}\Gamma(b_i)}\ {}_PF_Q\left[(a_P)\ ;\ (b_Q)\ ;\ -z\right]\ ,$$

it follows in particular from our lemma (by setting the parameters in accordance with the relation (2.12) and then finally letting $p\to 0$ in the resulting expression) that

$$(2.13)\quad {}_pD_\infty^\gamma\, g(0) = \frac{(-1)^\gamma}{a}\cdot \frac{\Gamma(1+\gamma)\prod_{i=1}^{P}\Gamma(1-a_i+\gamma)}{\prod_{i=1}^{Q}\Gamma(1-b_i+\gamma)}\int_0^\infty x^\gamma\, df(x)\ ,$$

which is our desired result (2.10).

Conversely, if ${}_pD_\infty^\gamma\, g(0)$ given by (2.10) exist, then it follows straight forwardly from the Fubini's theorem that

$$\int_0^\infty |x|^\gamma \, d\,f(x) < \infty \quad , \quad \gamma \, \varepsilon(-\infty, \infty) \, ,$$

and thus $f(x)$ possesses the absolute moment of γ^{th} order.

REMARK 3. By the elementary relation

$$(2.14) \quad H^{1,0}_{0,1}\left[z \,\Big|\, \begin{matrix} - \\ (0,1) \end{matrix}\right] = e^{-z} \, ,$$

the H-function transform (2.1) reduces to the Laplace-Stieltjes transform (1.1), and the relation (2.11) transforms into

$$(2.15) \quad \bar{\eta}_0(-,0;\gamma) = 1 \, .$$

In view of (2.14) and (2.15), Wolfe's result (1.6) follows at once from our Theorem 2.

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