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AN INTEGRAL EQUATION ASSOCIATED WITH ${}_{p+2}F_{p+1}$

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ABSTRACT

The solution of an integral equation associated with a generalized hypergeometric polynomial ${}_{p+2}F_{p+1}$ is obtained in terms of certain integral operators by the application of Rodrigues formula. The result given recently by Dixit (1978) is derived as a special case.

RESUMEN

La solución de una ecuación integral asociada con un polinomio

hipergeométrico generalizado se obtiene en términos de ciertos operadores integrales. Se emplea la fórmula de Rodrigues para conseguir el resultado, se deriva como caso especial, el resultado obtenido recientemente por Dixit (1978).

1. INTRODUCTION

An integral equation involving Legendre polynomial has been solved by Erdélyi (1963) by the application of Rodrigues' formula for Legendre polynomials. By following a similar technique integral equations possessing generalized Legendre polynomials and Jacobi polynomials as kernels have been obtained respectively by Singh, R.P. (1967) and Singh, C. (1970). Joshi (1974) has obtained the solution of an integral equation associated with Rice's polynomials. Dixit (1977) has derived the solution of generalized Rice's polynomials as its Kernel extending the work of Joshi. Recently Dixit (1978) has also given the solution of an integral equation whose Kernel is the hypergeometric polynomials ${}_4F_3$ thereby extending his earlier work.

In order to extend and unify the results on the various polynomials enumerated above, the authors have derived the solution of an integral equation associated with the polynomial ${}_{p+2}F_{p+1}$ in terms of certain integral operators with the help of Rodrigues' formula.

2. RODRIGUES FORMULA

It will be shown that

$$ {}_{p+2}F_{p+1}\left(\begin{matrix} -s, a_1, \ldots, a_p, 1+\gamma+\delta+s \\ b_1, \ldots, b_p, 1+\gamma \end{matrix} ; u\right) $$

$$ = \frac{u^{-\gamma}\,\Gamma(1+\gamma)}{\Gamma(1+\gamma+\delta+s)} \prod_{j=1}^{p} \left\{\frac{\Gamma(b_j)}{\Gamma(a_j)}\right\} \left\{ \left(\frac{d}{du}\right)^{\delta+s} u^{\gamma+\delta+s+1-b_p} \right\} $$

$$ \times \prod_{j=2}^{p} \left\{ \left(\frac{d}{du}\right)^{a_j-b_j} u^{a_j-b_{j-1}} \right\} \left(\frac{d}{du}\right)^{a_1-b_1} \left\{ u^{a_1-1}(1-u)^{s} \right\}, \tag{2.1} $$

To prove (2.1), we observe that

$$(\frac{d}{du})^{a_1-b_1}\{u^{a_1-1}(1-u)^s\} = (\frac{d}{du})^{a_1-b_1}\sum_{j=0}^{s}\frac{(-s)_j}{j!}u^{j+a_1-1}$$

$$= \sum_{j=0}^{s}\frac{(-s)_j(j+a_1-1)(j+a_1-2)\dots(j+b_1)u^{j+b_1-1}}{j!}$$

$$= \frac{\Gamma(a_1)}{\Gamma(b_1)}u^{b_1-1}\,{}_2F_1(-s,a_1;b_1;u) \qquad (2.2)$$

(2.2) is a known result (Sneddon, 1961,p.99).

Further it readily follows that

$$\{(\frac{d}{du})^{a_2-b_2}u^{a_2-b_1}\}(\frac{d}{du})^{a_1-b_1}\{u^{a_1-1}(1-u)^s\}$$

$$= \frac{\Gamma(a_1)}{\Gamma(b_1)}(\frac{d}{du})^{a_2-b_2}u^{a_2-1}\,{}_2F_1(-s,a_1;b_1;u)$$

$$\frac{\Gamma(a_1)\Gamma(a_2)}{\Gamma(b_1)\Gamma(b_2)}u^{b_2-1}\,{}_3F_2(-s,a_1,a_2;b_1,b_2;u) \qquad (2.3)$$

By the repeated application of (2.3), the result (2.1) follows.

(2.1) can be rewritten in the format

$${}_{p+2}F_{p+1}\left(\begin{matrix}-s,1+\gamma+\delta+s,a_1,\dots,a_p; \\ 1+\gamma,b_1,\dots,b_p;\end{matrix}\;\frac{u}{t}\right)$$

$$= t^{-s}u^{-\gamma}(\frac{d}{du})^{\delta+s}u^{\gamma+\delta+s+1-b_p}\frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma+\delta+s)}\prod_{j=1}^{p}\frac{\Gamma(b_j)}{\Gamma(a_j)}$$

$$\prod_{j=2}^{p}\{(\frac{d}{du})^{a_j-b_j}u^{a_j-b_{j-1}}\}(\frac{d}{du})^{a_1-b_1}\{u^{a_1-1}(t-u)^s\} \qquad (2.4)$$

In what follows C will denote the product

$$\frac{\Gamma(1+\gamma)\Gamma(b_1)\ldots\Gamma(b_p)}{\Gamma(1+\gamma+\delta+s)\Gamma(a_1)\ldots\Gamma(a_p)} . \tag{2.5}$$

3. THEOREM.

(i) If $s > 1$ and the function h is absolutely continuous on $(1,u_0)$ for some $u_0 > 1$;

(ii) γ, δ, a_j and b_j $\forall j, j=1,\ldots,p$ are all non-negative integers and $h(1)=0$, then the solution of the integral equation

$$\int_1^u {}_{p+2}F_{p+1}\left(\begin{matrix}-s,1+\gamma+\delta+s,a_1,\ldots,a_p\\ 1+\gamma,b_1,\ldots,b_p\end{matrix};\frac{u}{t}\right)K(t)dt=h(u), \tag{3.1}$$

for $1 < u > u_0$ is given by

$$K(u)= \frac{(-1)^s Cu^s}{s\,!}\left(\frac{d}{du}\right)^{s+1}\theta(u). \tag{3.2}$$

PROOF. Applying (2.1) to (3.1) and interchanging the operators of integration and differentiation, which is justified, we obtain

$$\left(\frac{d}{du}\right)^{\delta+s}\int_1^u u^{\gamma+\delta+s-b_p+1}\left\{\left(\frac{d}{du}\right)^{a_p-b_p}u^{a_p-a_{p-1}}\right\}$$

$$\left\{\left(\frac{d}{du}\right)^{a_{p-1}-b_{p-1}}u^{a_{p-1}-a_{p-2}}\right\}\ldots\left[\left(\frac{d}{du}\right)^{a_1-b_1}\left\{u^{a_1-1}(t-u)^s\right\}\right]t^{-s}K(t)dt$$

$$= Cu^{\gamma}h(u). \tag{3.3}$$

The successive integration $(\delta+s)$ times and interchanging the operators of integration and differentiation yields

$$\left(\frac{d}{du}\right)^{a_p-b_p} \int_1^u u^{a_p-b_{p-1}} \{ \left(\frac{d}{du}\right)^{a_{p-1}-b_{p-1}} u^{a_{p-1}-b_{p-2}} \}$$

$$\ldots\{ \left(\frac{d}{du}\right)^{a_1-b_1} u^{a_1-s} (t-u)^s \} t^{-s} K(t)dt$$

$$= \frac{Cu^{b_p-\gamma-\sigma-1}}{(\sigma-1)!} \int_1^u (u-v)^{\sigma-1} H(v)dv, \qquad (3.4)$$

where $H(v)=u^{\gamma}h(u), \sigma=\delta+s$.

we now introduce an integral operator denoted by I,

$$I\left[\rho:\Phi(u)\right] = \frac{1}{(\rho-1)!} \int_1^u (u-v)^{\rho-1} \Phi(v)dv. \qquad (3.5)$$

Hence

$$\left(\frac{d}{du}\right)^{\rho} I\left[\rho:\Phi(u)\right] = \Phi(u) \qquad (3.6)$$

and $I\left[\rho:\Phi(u)\right]=0$ for $u=1$.

In the contracted form, we write

$$I\left[a_r-b_r;\Phi(u)\right] = I_r\left[\Phi(u)\right]. \qquad (3.7)$$

(3.4) can now be rewritten as

$$\left(\frac{d}{du}\right)^{a_p-b_p} \int_1^u u^{a_p-b_{p-1}} \{ \left(\frac{d}{du}\right)^{a_{p-1}-b_{p-1}} u^{a_{p-1}-b_{p-2}} \}$$

$$\ldots \left[\left(\frac{d}{du}\right)^{a_1-b_1} \{ u^{a_1-1}(t-u)^s \}\right] t^{-s} K(t)dt$$

$$= Cu^{b_p-\gamma-\sigma-1} I\left[\sigma:H(u)\right]. \qquad (3.8)$$

Integrating successively (a_p-b_p) times and interchanging the order of integration and differentiation, we find that

$$(\frac{d}{du})^{a_{p-1}-b_{p-1}} \int_1^u u^{a_{p-1}-b_{p-2}} \{ (\frac{d}{du})^{a_{p-2}-b_{p-2}} u^{a_{p-2}-b_{p-3}}\}$$

$$\ldots (\frac{d}{du})^{a_1-b_1} \{u^{a_1-1} (t-u)^s\} t^{-s} K(t)dt$$

$$= Cu^{b_{p-1}-a_p} I\left[a_p-b_p : u^{b_p-\sigma-s-1} I(\sigma:H(u))\right]. \qquad (3.9)$$

Proceeding in a similar manner and employing the operators I_j for $j=p-1,\ldots,2,1$ it readily follows that (3.9) transforms into the form

$$\int_1^u (t-u)^s t^{-s} K(t)dt = C\theta(u), \qquad (3.10)$$

where

$$\theta(u) = (u^{1-a_1} I_1) \mid (u^{b_1-a_2} I_2)(u^{b_2-a_3} I_3)\ldots$$

$$\times u^{b_{p-1}-a_p} I_p\left[u^{b_p-\sigma-s-1} I(\sigma:H(u))\right]. \qquad (3.11)$$

Finally differentiating s+1 times with respect to u, we arrive at the result (3.2).

For p=2, (3.2) reduces to a result given recently by Dixit (1978).

4. CONSIDER THE INTEGRAL EQUATION

$$\int_1^u (\frac{u}{t})^\rho \; {}_{p+2}F_{p+1}\left[\begin{matrix} -s, a_1, \ldots, a_p, \gamma+\delta+s+1; \\ 1+\gamma, b_1, \ldots, b_p; \end{matrix} \; \frac{u}{t}\right] K_1(t)dt$$

$$= h_1(u), \qquad \ldots. \qquad (4.1)$$

for $1 < u < u_0$, which reduces to (3.1), when $\rho=0$.

However, if we set $t^{-\rho}K_1(t)=K(t)$, and $t^{-\rho}h_1(t)=h(t)$, (4.1) again reduces to (3.1) and its solution is given by

$$K_1(t) = \frac{(-1)^s}{s!} Cu^{s+\rho}\left(\frac{d}{du}\right)^{s+1}\theta(u), \tag{4.2}$$

where $\theta(u)$ is defined in (3.11).

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