Rev.Téc.Ing., Univ.Zulia
Vol.4, N°s.1 y 2, 1981

RECENT RESULTS ON (r,λ) - DESIGNS AND RELATED CONFIGURATIONS

(Recibido el 15 de Junio de 1978)

M. Deza
C.N.R.S.
París - France

R.C. Mullin
Department of Statistics
University of Waterloo
Ontario - Canada

S.A. Vanstone
St. Jerome's College
University of Waterloo
Ontario - Canada

ABSTRACT

In recent times (r,λ) - designs and related configurations have been discussed by many authors in many contexts. Because these configurations have both mathematical and practical interest, it is the purpose of the present paper to bring together several of these results. Related configurations include the Δ - systems of Erdos and Rado, and balanced equidistant codes and equidistant permutation arrays.

RESUMEN

Recientemente han sido discutidos por varios autores en diferentes contextos, diseños (r,λ) y configuraciones relacionadas. Como estas configuraciones tienen interés matemático y también práctico, el objeto de este trabajo es unificar algunos de estos resultados. Configuraciones relacionadas incluyen el sistema - Δ de Erdos y Rado, códigos equidistantes balanceados y arreglos permutacionales equidistantes.

1. INTRODUCTION

The designs discussed in this paper arise naturally in many areas of combinatorial theory, especially coding theory and combinatorial design theory. In the case $\lambda = 1$, they are a subclass of the class of pairwise balanced designs, the latter being a central tool of design theory. Although a thorough discussion of this subject is beyond the scope of the present survey, the reader is referred to the works of Wilson [41], [42], for fundamental results in this area. Although there are many other papers on this topic, references to these are omitted since our main concern is with (r,λ)-systems. These systems have also arisen in the theory of balanced equidistant codes, see [1], [29], etc. Some of the results cited in this survey are basic to the establishment of good bounds for the dictionary size of such codes. Further, the use of (r,λ)-systems in multiplexing schemes has recently been investigated. Moreover it has been shown recently [20] that the more general class of (r,λ)-systems is required to yield the extremal configurations for the Doehlert-Klee problem [13]; prior to this it appeared that the subclass of (r,λ)-designs known as BIBD's (discussed below) contained the required configurations.

2. PRELIMINARIES

An (r,λ)-design $\mathcal{D}$ (regular pairwise balanced design) is a system consisting of a finite set V of elements (called varieties) and a collection B of subsets of V (called blocks) such that

(1) every pair of distinct varieties is contained in precisely λ blocks.

(2) every variety is contained in exactly r blocks.

Subsequently, we let v,b,n denote the number of varieties, number of blocks and $r - \lambda$ respectively.

A balanced incomplete block design (BIBD) is an (r,λ)-design in which every block has the same cardinality (size) k and for which $r > \lambda > 0$. The numbers (v,b,r,k,λ) are called the parameters of the block design. BIBD's have received a great deal of attention and there is extensive literature on the subject. Again, a detailed survey of BIBD's is beyond the scope of a single paper and so we restrict ourselves to the more general (r,λ)-designs. It should be noted that the deletion of a t-subset of varieties from a BIBD with parameter set (v,b,r,k,λ) gives an (r,λ)-design on $v-t$ varieties. It is not true however that every (r,λ)-design is obtainable in this manner. An example of the latter type of design can be found in [30].

H.J. Ryser [29] has shown that for any (r,λ)-design $\mathcal{D}$ if $b = v$ then $\lambda(v-1) = r(r-1)$ and that $\mathcal{D}$ is a BIBD with block size r. From this or otherwise it can be shown that for any (r,λ)-design, $b \geq v$. An (r,λ)-design is called elliptic, parabolic or hyperbolic accordingly as the expression $\lambda(v-1) - r(r-1)$ is negative, zero or positive. An (r,λ)-design $\mathcal{D}$ is said to be reducible if $\mathcal{D}$ contains a block containing all varieties (called a complete block) or a set of v blocks each of size one whose union is V (called a complete set of singletons). If $\mathcal{D}$ is not reducible then it is irreducible. It was shown in [30] that all irreducible designs with $\lambda = 1$ are elliptic or parabolic. This is not true in general for $\lambda > 1$. This will be discussed in greater detail in section 5.

Let $\mathcal{D}$ be an (r,λ)-design defined on the variety set V. $\mathcal{D}'$ is called a restriction of $\mathcal{D}$ to V' if $V' \subset V$ and $\mathcal{D}'$ is obtained from $\mathcal{D}$ by deleting the varieties of V/V' from the blocks of $\mathcal{D}$. We will introduce other definitions as they are required.

3. EMBEDDINGS

By definition, an (r,λ)-design $\mathcal{D}$ is embeddable in an (r,λ)-design $\mathcal{D}'$ if $\mathcal{D}$ is isomorphic to some restriction of $\mathcal{D}'$. The first result we cite concerns the embeddability of a BIBD with parameters (v,b,r,k,λ) in an (r,λ) design. The proof of Theorem 2.1 appears in

[30].

THEOREM 3.1 *A BIBD with parameters (v,b,r,k,λ) can be embedded in an (r,λ)-design only if k divides $r-\lambda$.*

An (r,λ)-design is called trivial if it contains λ complete blocks ; otherwise , it is called non-trivial. For any non-trivial $(r,1)$-design , it can be shown that the maximum number of varieties is n^2+n+1, and any non-trivial $(r,1)$-design attaining this is a finite projective plane of order n. What can be said about embedding $(r-1)$-designs into these extreme configurations? The following two theorems provide some results in this area.

THEOREM 3.2 *If D is a non-trivial $(r,1)$-design having $v \leq n^2+n$ varieties and $b \leq n^2+n+1$ blocks and D contains a block of size $r-1$ then D is embeddable in an $(r,1)$-design on $v+1$ varieties.*

If the number of varieties v in a non-trivial $(r,1)$-design D is such that $n^2 \leq v \leq n^2+n$ then a sharper result is possible. This is stated as

THEOREM 3.3 *If D is a non-trivial $(r,1)$-design on v varieties where $n^2 \leq v \leq n^2+n$ then D is embeddable in a finite projective plane of order n. (i.e., a BIBD with parameters $(n^2+n+1,\ n^2+n+1,\ n+1,\ n+1,\ 1)$.*

A proof of Theorems 3.2 and 3.3 can be found in [39]. Theorem 3.3 has recently been strengthened; it has been shown [22] that if D is a non-trivial $(r,1)$-design having $v \geq n^2-\alpha$ varieties where $\alpha < \sqrt{n/2}$ then D is embeddable in a finite projective plane of order n.

A particularly important class of design in embedding theorems is the $(2n,n)$-designs. For such designs, we have the following theorems.

THEOREM 3.4 *If $\mathcal{D}$ is a non-trivial $(2n,n)$-design having v varieties where $n^2 + 1 \leq v \leq n^2 + n$ then $\mathcal{D}$ is embeddable in a non-trivial $(2n,n)$-design having $n^2 + n + 1$ varieties.*

It was shown by J.H. Van Lint [31] that the existence of a non-trivial $(2n,n)$-design on $n^2 + n + 1$ varieties implies the existence of a finite projective plane of order n. A generalization of this result appears in [16] and [36]. We state it as

THEOREM 3.5 *There exists a non-trivial $(2n,n)$-design having $v = n^2 + n + 1 - \alpha$ varieties where $\alpha \leq 2n - 4$ iff there exists a non-trivial $(n+1,1)$-design having v varieties.*

This result was independently proved by J.I. Hall [16] where he shows it is true for $a \leq \frac{n^2-2n-2}{2}$.

It is clear that Theorem 2.5 will provide us with a non-existence result for $(2n,n)$-designs when no finite projective plane of order n exists. As was mentioned earlier, the $(2n,n)$-designs are a very important class of (r,λ)-designs. Any (r,λ)-design $\mathcal{D}$ on v varieties implies the existence of a $(2n,n)$-design on $v - 1$ varieties. Because of this relationship one might expect that there is an embedding theorem for (r,λ)-designs $(\lambda > 1)$ similar to Theorem 2.4. This is not in general true. In [37], it is shown that there exist non-trivial (r,λ)-designs on $n^2 + n$ varieties which cannot be embedded in any (r,λ)-design on $n^2 + n + 1$ varieties.

4. UPPER BOUNDS

Define the function $v_0(r,\lambda)$ to be the smallest positive integer such that if $v > v_0(r,\lambda)$ then the only (r,λ)-designs on v varieties are trivial. This function was first introduced by V. Chvátal [4]. A related function introduced in [15] is $v_1(r,\lambda)$, is the smallest positive integer such that if $v > v_1(r,\lambda)$ then the only (r,λ)-designs on v varieties are reducible. Finally, $v_p(r,1)$ which appears in

[25] is the largest positive integer such that there exists a non-trivial $(r,1)$-design $\mathcal{D}$ on $v_p(r,1)$ varieties in which the number of blocks in $\mathcal{D}$ is less than or equal to $n^2 + n + 1$.

Little is known about $v_1(r, \;)$ and $v_p(r,1)$. However, a good upper bound for $v_0(r,\lambda)$ is known and the following inequalities hold.

$$v_p(r,1) \leq v_1(r,1)$$

and

$$v_1(r,\lambda) \leq v_0(r,\lambda).$$

Staton and Mullin [30] gave the following bound for $v_0(r,1)$.

THEOREM 4.1 *For any positive integer r*

$$v_0(r,1) \leq n^2 + n + 1.$$

Applying the strengthened version of Theorem 3.3, it is possible to improve theorem 4.1.

THEOREM 4.2 *For any positive integer r*

(i) $v_0(r,1) = n^2 + n + 1$ *if n is the order of a finite projective plane.*

(ii) $v_0(r,1) \leq n^2 - \alpha$ *if* $\alpha < \sqrt{n/2}$ *and n is not the order of a finite projective plane.*

The first case of interest for the function $v_0(r,1)$ occurs when $n = 6$, since it is well known that no finite projective plane of order 6 exists. It has been shown [27], [18] that $v_0(7,1) = 31$. The only example known of a non-trivial $(7,1)$-design on 31 varieties is obtained by adding a complete set of singletons to the finite projective plane of order 5. It has been shown ([23], [24], [25], [26]) that this is the only way of obtaining a non-trivial $(7,1)$-design on 31 varieties. Thus $v_1(7,1) < 31$. It is shown in [26] that $25 \leq v_p(7,1) \leq 28$, and thus if there exists a non-trivial irreducible $(7,1)$-design on 30 varieties it must contain at least 44 blocks.

For $\lambda > 1$, the following result on $v_0(r,\lambda)$ appeared in [28].

THEOREM 4.3 *For positive integers r and λ, such that*

$$\lambda \geq n^2 + n - 1, \quad v_0(r,\lambda) = \lambda + 2.$$

The (r,λ)-designs which have $\lambda \geq n^2 + n - 1$ and $v = \lambda + 2$ have been completely characterized [28]. They have block sizes of 1, $v-1$ and v only and thus are called near-trivial. This notion of near trivial has been generalized [6] to z-trivial designs. A z-trivial (r,λ)-design has only block sizes

$$1, 2, \ldots, \left[\frac{z}{2} + 1\right], v - z, v \equiv z + 1, \ldots, v - 1, v.$$

Clearly a 0-trivial design is trivial, a 1-trivial is near-trivial and any z-trivial is a $z + i$-trivial design for i a positive integer. A few results on z-triviality have been obtained [6] but this concept has been by no means explored fully. The only designs satisfying the hypothesis of Theorem 4.3 are either 0-trivial or 1-trivial.

Theorem 4.3 and a fundamental result in [5] concerning block sizes of $(2n,n)$-designs leads to

THEOREM 4.4 *For any positive integers* r *and* λ $(r > \lambda)$

$$v_0(r,\lambda) \leq \max\{\lambda + 2,\ n^2 + n + 1\}.$$

A proof of this appears in [40]. Recently this result has been improved ([22]).

THEOREM 4.5 *For any positive integers* r *and* λ $(r > \lambda)$

(i) $v_0(r,\lambda) = \lambda + 2$ *if* $\lambda \geq n^2 + n - 1$.

(ii) $v_0(r,\lambda) = n^2 + n + 1$ *if* $\lambda < n^2 + n - 1$ *and* n *is the order of a finite projective plane.*

(iii) $v_0(r,\lambda) \leq \max\{\lambda + 2,\ n^2 - 1\}$ *if* n *is not the order of a finite projective plane.*

As was mentioned earlier, little is known about the functions $v_1(r,\lambda)$ and $v_p(r,1)$. One result on $v_1(r,\lambda)$ appears in [37].

THEOREM 4.6 *Let* $n = r - \lambda$. *Then*

$$v_1(r,\lambda) = n^2 + n + 1$$

iff n *is the order of a finite projective plane and* r *is equal to one of* n^2, $2n$ *or* $n + 1$.

Other results on $v_1(r,\lambda)$ can be found in ([15], [28]).

5. (r,λ)-DESIGNS AND CODES

For a much broader study of the ideas in this section the reader is referred to [1]. A block code of length n, size N and distance d over an alphabet A of q symbols is a collection of vectors with entries from A of length n such that the Hamming distance (the number of components in which two vectors differ) between any two vectors is at least d. N is the number of vectors in C. A fundamental result in this area is the Plotkin bound. (Theorem 5.1).

THEOREM 5.1 *If a block code C of length n, size N and distance d exists, then*

$$d \leq \frac{nN(q-1)}{(N-1)q}$$

Let $\mathcal{D}$ be an (r,λ)-design $\mathcal{D}$ having v varieties and b blocks. Define the $v \times b$ matrix (incidence matrix of $\mathcal{D}$)

$$A = [a_{ij}] \quad \text{where} \quad a_{ij} = \begin{cases} 1 \text{ if } V_i \in B_j \\ 0 \text{ otherwise.} \end{cases}$$

One can consider the rows of A as binary codewords of a block code of length b and size v. The distance between any two codewords is precisely $2(r,\lambda)$ and every codeword contains precisely r ones. Such codes are called equidistant-equiweight block codes and are equivalent to the incidence matrix of (r,λ)-designs. Many results on (r,λ)-designs have been given using the notation of equidistant codes. The reader is referred to [6], [16], [17], [31].

J. Hall [17], using the equidistant-equiweight code approach, was able to settle a conjecture of Stanton-Mullin. Recall from section 1 that Stanton and Mullin had shown that any $(r,1)$-design was either elliptic or parabolic. They conjectured that this was also true for any (r,λ)-design which was irreducible. Hall proved the conjecture for $\lambda = 2$ and found counterexamples for $\lambda \geq 3$.

The proof of theorem 4.4 relies heavily on an inequality for bounding the block sizes in $(2n,n)$-designs. This inequality came as a result of studying equidistant codes [5]. One has the advantage of using the Hamming distance properties of the code which is not an evident property of the corresponding (r,λ)-design.

Before ending this section, we mention that if equality holds in the Plotkin bound of Theorem 5.1, then it can be shown that the code can be used to produce a very particular type of BIBD. The reader is referred to [1] for details.

6. Δ-SYSTEMS

A strong Δ-system is a set of subsets $\{S_1, S_2, \ldots, S_n\}$ from a finite set such that $|S_i| = t$ for each i and such that there exists a set L with the property that $S_i \cap S_j = L$ for all $i \neq j$. The following theorem is due to Erdös and Rado [14].

THEOREM 5.1 *There exists a function* $\phi(m,t)$ *such that every family* $S_1, S_2, \ldots, S_\phi$ *of sets with* $|S_i| = t$ *contains a strong Δ-system having more than* m *sets.*

Erdös and Rado showed that $m^t < \phi(m,t) < t!\, m^t$.
This was later improved by Chvátal [4] and by Abbot and Hanson to

$$\phi(m,t) < \frac{(t+1)!\, m^t}{2^t}$$

Chvátal deduced the Erdös-Rado theorem from Ramsey's theorem |4| by introducing the idea of a weak Δ-system. A weak Δ-system is a collection of subsets $S_1, S_2, \ldots, S_m$ from a finite set such that $|S_i| = t$ for each i and such that there exists an integer ℓ with the property that $|S_i \cap S_j| = \ell$ for all $i \neq j$. As an example, every finite projective plane is a weak Δ-system but not a strong Δ-system. Clearly, every strong Δ-system is a weak Δ-system but the converse is by no means true . The converse is true , however , when the number of subsets in the weak Δ-system is large. If one considers the so called dual of a weak Δ-system (i.e., let the subsets be elements and elements be subsets such that an element is in a subset if the original subset contained the element associated with the new subset). Then one obtains an (r,λ)-design with $r = t$ and $\lambda = \ell$. Hence, all of the results of sections 2 and 3 are applicable to weak Δ-systems.

We end this section by mentioning that P. Erdös conjectures that

$$\phi(m,t) < (Cm)^t$$

for some absolute constant C. He offers a prize of one thousand dollars to anyone who can settle the question.

7. SOME GENERALIZATIONS OF $v_0(r,\lambda)$.

The basis of sections 2 and 3 is the study of the function $v_0(r,\lambda)$. Recall that it is the smallest positive integer such that if $v > v_0(r,\lambda)$ then the only (r,λ)-designs on v varieties are trivial. Below we will give a number of generalizations and specifications of this function.

(I) Instead of an (r,λ)-design , suppose we consider a collection B of blocks from a v-set V such that every variety of V is contained in r blocks of B and any distinct unordered pair of varieties is contained in λ blocks where $\lambda \in \{\lambda_1, \lambda_2, \ldots, \lambda_s\}$ and $\lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_s < r$. We call such a system an $(r,\{\lambda_1, \lambda_2, \ldots, \lambda_s\})$-design. These designs have been studied in [10] and [11]. When $s = 1$ it is clear that we have an (r,λ)-design. In what follows we restrict our discussion to the case $s = 2$ since it is quite different from $s = 1$ and gives the flavour of the more general situation.

Let $\mathcal{D}$ be an $(r,\{\lambda_1, \lambda_2\})$-design having b blocks. By results of Deza [11], we know that there exist minimal functions C_1, C_2, C_3 and C_4 of r, λ_1 and λ_2 such that

(a) If $v > C_1 b$, then $\mathcal{D}$ contains λ_1 complete blocks. Notice that in the case of an (r,λ)-design the same phenomena occurs but that the size of v did not depend on b.

(b) If $v < C_2 b$ then $v \leq \frac{(b-\lambda_1)(b-\lambda_2)}{(r-\lambda_1)(r-\lambda_2)}$ In the case when we have an (r,λ)-design if $v > v_0(r,\lambda)$ then $v = \frac{b-\lambda}{r-\lambda}$.

(c) If $v > C_3 b$, let $B_1, B_2, \ldots, B_r$ be the blocks containing a variety x. Then $B_1/\{x\}, B_2/\{x\}, \ldots, B_r/\{x\}$ is a trivial (λ_1,λ_2)-design having r blocks and $\frac{r-\lambda_1}{\lambda_2-\lambda}$ varieties. Again, in the case of an (r,λ)-design $v_0(r,\lambda)$ takes the role of $C_3 b$. As a consequence of the above, we have that $(\lambda_2 - \lambda_1) | (r - \lambda_1)$.

(d) If $v > C_4 b$ then there exist $x,y \in V$ $(x \neq y)$ and $x',y' \in V$ $(x' \neq y')$ such that x,y are contained in λ_1 blocks and x',y' are contained in λ_2 blocks of $\mathcal{B}$. In other words, for V sufficiently large both λ_1 and λ_2 must be realized for some pairs of varieties. This property has no meaning in the case of an (r,λ)-design.

As was indicated above, the four functions, $C_1 b$, $C_2 b$, $C_3 b$ and $C_4 b$ all collapse to $v_0(r,\lambda)$ in the case of an (r,λ)-design. Good estimations of $v_0(r,\lambda)$ are known but little can be said about C_1, C_2, C_3 and C_4 except that they are well defined.

Few results on the above generalizations are known even in the case of $s = 2$. One does not even have an estimation of their magnitudes relative to each other. However, in [11] the following was obtained as a generalization of the Fisher inequality for (r,λ)-design as applied to $(r, \{0, \lambda_2\})$-designs. It was shown that if $b > v$ in an $(r, \{0, \lambda_2\})$-design then $\lambda_2 | r$. In this case, we see that $C_3 = 1$ for an $(r, \{0, \lambda_2\})$-design. The dual of the locally symmetric designs of Cameron [3] give an example of an $(r, \{0, \lambda_2\})$-design with $v > b$ but not having the property that the design formed from the blocks containing a given variety is a trivial $(\lambda_2, 0)$-design. Other examples of $(r, \{0, \lambda_2\})$-designs are the dual of a BIBD (this is a $(k,\{0,1\})$-design) and the dual of any affine resolvable design.

One further generalization is in the following area. We call a collection of blocks B from a v-set V, an $(r, \lambda_1, \lambda_2)$-design if every variety occurs in r blocks of B, every distinct unordered pair of varieties in λ_1 blocks and every distinct unordered 3-set occurs in λ_2 blocks. If we do not require the condition for λ_1 then we write $(r, -, \lambda_2)$-design. It is clear that the condition on λ_2 does not imply the condition on λ_1. The above definitions can easily be extended to $\lambda_1, \lambda_2, \ldots, \lambda_t$ if desired. From remark 5 of [6] and Theorem 9 of [11] it follows that there exist minimal functions $C_1(r,\lambda_1,\lambda_2)$ and $C_2(r,\lambda_2)$ such that

(a) If $v > C_1$ then any $(r, \lambda_1, \lambda_2)$-design contains λ_2 complete blocks. Deza and Frankl [7] conjecture that $C_1 = O(r)$. Vanstone [35] has shown that $C_1 = O(r-\lambda)$.

(b) If $v > C_2$ then any $(r, -, \lambda_2)$-design has λ_2 complete blocks. An interesting question is whether or not $v_0(r,\lambda) > C_2(r,\lambda_2)$.

8. EQUIDISTANT PERMUTATION ARRAYS

In this section, we present another closely related problem which was introduced by D.W. Bolton [2].

An equidistant permutation array (E.P.A.) is a $v \times r$ array in which every row is a permutation of the integers $1, 2, \ldots, r$ and such that any two distinct rows of the array have precisely λ columns in common. We denote such an E.P.A. by $A(r,\lambda;v)$. Define $R(r,\lambda)$ to be the largest value of V for which there exists an $A(r,\lambda;v)$.

A resolution R of an (r,λ)-design D is a partitioning of the blocks into classes (called resolution classes) $R_1, R_2, \ldots, R_r$ such that each variety of D is contained in precisely one block of each R_i, $1 \leq i \leq r$. An (r,λ)-design D is called orthogonal if there are two resolutions R, R' of D such that any resolution class of R has at most one block in common with any resolution class of R'. The following result appears in [9].

THEOREM 6.1 *There exists an $A(r,\lambda;v)$ iff there exists an orthogonal (r,λ)-design having v varieties.*

As a consequence of this, the upper bounds of section 3 on $v_0(r,\lambda)$ can be applied to $R(r,\lambda)$. The first general bounds for $R(r,\lambda)$ where obtained in [6]. It should be noted that orthogonal (r,λ)-designs form a very particular class of (r,λ)-designs and hence in many cases the upper bound for $v_0(r,\lambda)$ may not be a particularly good bound for $R(r,\lambda)$. For instance, it can be shown [8] that

$$R(r,1) \le r(r-3).$$

However, it should also be mentioned that

$$R(\lambda+3,\lambda) = \lambda+2 \quad \text{for all } \lambda \ge 12.$$

A number of results concerning E.P.A.s have been obtained. The interested reader if referred to [6], [8], [9], [19] and [38] in the bibliography.

One can generalize the idea of an E.P.A. by asking that any two distinct rows of the array have at least (at most) λ columns in common. Analogously one can define $R(r, \ge\lambda)$ $\left(R(r, \le\lambda)\right)$ to be the maximum number of rows in such an array. Such arrays have been investigated ([6],[7]).

9. SOME OPEN PROBLEMS

We list a number of problems which remain open.

1. What are the values of $v_1(7,1)$ and $v_p(7,1)$?
2. Is $v_0(r,\lambda) = v_0(n+1,1)$ for small λ where $n = r - \lambda$?
3. If n is not the order of a finite projective plane then, is $v_0(n+1,1) = q^2 + q + 1$ where q is the largest integer less than n for which there exists a finite projective plane of order q?
4. Is $\phi(m,t) < (C_m)^t$ for some absolute constant C?
5. Let T be a set of resolutions of an (r,λ)-design with the property that any two resolutions of T are orthogonal. It is known

[8] that $|T| \leq r - \lambda$. Is this the best possible?

6. Find good estimates for the functions $v_0(r,\lambda)$, $v_1(r,\lambda)$, $v_p(r,1)$, $R(r,\lambda)$, $R(r,\leq\lambda)$, $R(r,\geq\lambda)$, C_1, C_2, C_3 and C_4.

REFERENCES

[1] BLAKE, I.F. and MULLIN, R.C., *The Mathematical Theory of Coding*, Academic Press (1976).

[2] BOLTON, D.W., Private communication.

[3] CAMERON, P., *Locally Symmetric Designs*, Geometricae Dedicata 3(1974), 65-76.

[4] CHVATAL, V., *On Finite Δ-systems of Erdos and Rado*, Acta Math. Acad. Sc. Hungaricae, 21 (1970), 341-355.

[5] DEZA, M., *Une Propriete Extremale des Plans Projectifs Finis dans une Classe de Codes Equidistants*, Discrete Math 6(1973), 343-352.

[6] DEZA, M., *Matrices Dont Deux Lignes Quelconques Coincident dans un Nombre Donne de Positions communes*, Journal of Combinatorial Theory, Series A 20-3 (1976), 306-318.

[7] DEZA, M. and FRANKL, P., *On the Maximum Number of Permutations with Given Maximal or Minimal Distance*, Journal of Combinatorial Theory, Series A (to appear).

[8] DEZA, M., MULLIN, R.C. and VANSTONE, S.A., *Orthogonal Systems*, Aequationes Math (to appear).

[9] DEZA, M., MULLIN, R.C. and VANSTONE, S.A., *Room Squares and Equidistant Permutation Arrays*, Ars Combinatoria 2(1976), 235-244.

[10] DEZA, M., ERDOS, P. and FRANKL, P., *Intersection Properties of Systems of Finite Sets*, Proceedings of the fifth Hungarian Conference on Combinatorics, 1976 (to appear).

[11] DEZA, M., *Generalized Perfect Packing as a Generalization of Matroid-designs and Simple t-designs*, Proceedings of Paris-Orsay Conference on Combinatorics, 1976 (to appear).

[12] DEZA, M., ERDOS, P. and SINGHI, N., *Combinatorial Problems on Subsets and their Intersections*, Advances in Math. (to appear).

[13] DOEHLERT, D.H. and KLEE, V.L., *Experimental Designs through Level Reduction of the d-dimensional Cuboctahedron*, Discrete Math., 2 (1972), 309-334.

[14] ERDOS, P. and RADO, R., *Intersection Theorems for Systems of Sets*, Journal of London Math. Society, 35 (1960), 85-90.

[15] GARDNER, B.I., *On Coverings and* (r,λ)*-systems*, Ph.D. thesis, University of Waterloo, (1972).

[16] HALL, J., *Bounds for Equidistant Codes and Partial Projective Planes*, Discrete Math. 17 (1) (1977), 85-94.

[17] HALL, J., *On two Conjectures of Stanton and Mullin*, Journal of Combinatorial Theory, Series A, (to appear).

[18] HALL, J.I. , JANSSEN, A.J.E.M., KOLEN, A.W.J. and Van LINT , J.H., *Equidistant Codes with Distance 12*, Discrete Math., 17 (1) (1977), 71-83.

[19] HEINRICH, K. and Van REES, G.H.J. , *Equidistant Permutation Arrays of Index One*, (to appear).

[20] McCARTHY, D., STANTON, R.G. and VANSTONE, S.A., *An Extremal Class of* (r,λ)*-designs Related to a Problem of Doehlert and Klee*, Ars Combinatoria (1976), 305-317.

[21] McCARTHY, D. and VANSTONE, S.A., *On the Structure of* (r,λ)*-designs*, Discrete Math., (submitted).

[22] McCARTHY, D. and VANSTONE, S.A., *Embedding* $(r,1)$*-designs in Finite Projective Planes*, Discrete Math., (to appear).

[23] McCARTHY, D., MULLIN, R.C., SCHELLENBERG, P.J., STANTON, R.G. and VANSTONE, S.A., *The Nonexistence of* $(7,1)$*-designs:* $v = 31$, $b \geq 50$, Proceedings of the fifth Conference on Numerical Math., University of Manitoba (1975), 497-532.

[24] __________, *The Nonexistence of* $(7,1)$*-designs:* $v = 31$, $b \leq 43$, Proceedings of the fifth Conference on Numerical Math., University of Manitoba (1975), 479-496.

[25] __________, *Towards the Nonexistence of (7,1)-designs with 31 Varieties*, Proceedings of the sixth Conference on Numerical Math., University of Manitoba (1976), 265-285.

[26] __________, *An Approximation to a Projective Plane of Order 6*, Ars Combinatoria 2 (1976), 169-189.

[27] MULLIN, R.C., and VANSTONE, S.A., *On Regular Pairwise Balanced Designs of Order 6 and Index One*, Utilitas Math. 8, (1975) , 349-369.

[28] MULLIN, R.C., *An Asymptotic Property of (r,λ)-systems*, Utilitas Math., 3 (1973), 139-152.

[29] RYSER, H.J., *Combinatorial Mathematics*, The Carus Mathematical Monographs, (1963).

[30] STANTON, R.G. and MULLIN, R.C., *Inductive Methods for Balanced Incomplete Block Designs*, Ann. Math. Statist., 37 (1966), 1348-1354.

[31] Van LINT, J.H., *A Theorem on Equidistant Codes*, Discrete Math. 9, (1973), 353-358.

[32] Van LINT, J.H., *Equidistant Point Sets*, Proceedings British Combinatorics Conference, Aberdeen London Math. Society, University Press, 1974.

[33] VANSTONE, S.A., *The Structure of (r,λ)-designs*, Ph.D. thesis, University of Waterloo, (1974).

[34] VANSTONE, S.A. and SCHELLENBERG, P.J., *Some Recursive Constructions for Equidistant Permutation Arrays*, Journal of Combinatorial Theory, Series A, (submitted).

[35] VANSTONE, S.A., private communication.

[36] VANSTONE, S.A. and McCARTHY, D., *(r,λ)-designs and Finite Projective Planes*, Utilitas Math., 11 (1977), 57-72.

[37] VANSTONE, S.A., *Extremal (r,λ)-designs*, Discrete Math. (submitted).

[38] VANSTONE, S.A. and SCHELLENBERG, P.J., *Some Results on Equidistant Permutations*, 6th Numerical Math. Conference, University of Manitoba, (1976), 389-410.

[39] VANSTONE, S.A., *The Extendibility of $(r,1)$-designs*, Proceedings of the third Conference on Numerical Math., University of Manitoba (1973), 409-418.

[40] VANSTONE, S.A., *A Bound for $v_0(r,\lambda)$*, Proc. 5th S-E Conference Combinatorics, Graph Theory and Computing, Boca Raton (1974), 661-673.

[41] WILSON, R.M., *An Existence Theory for Pairwise Balanced Designs, II*, Journal of Combinatorial Theory, Series A. 13 (1972), 246-273.

[42] WILSON, R.M., *An Existence Theory for Pairwise Balanced Designs, III*, Journal of Combinatorial Theory, Series A. 18 (1975), 71-79.

[43] De WITTE, P., *On the Embeddability of Linear Spaces in Projective Planes of Order n*, (to appear).