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ELASTO-PLASTIC STRESSES IN ANISOTROPIC NON-HOMOGENEOUS CYLINDER

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D. Ghosh
Department of Mathematics
University of Ife
Ile-Ife - Nigeria

ABSTRACT

Present investigation is concerned with the elasto-plastic solution of an anisotropic non-homogeneous hollow cylinder. Power law variation has been assumed for all material constant in elastic stage and also for parameters characteristic of the state of anisotropy in plastic yielding. It is observed that plastic yielding may start at the inner surface of the cylinder or at the outer surface and there is a possibility of forming more than one plastic zone depending on the value of constants of the particular material as has been seen in the case of baryte. Numerical results showing distributions of stresses and strains and how the plastic yielding starts in the boron/epoxy cylinder are presented in tabular form.

RESUMEN

La presente investigación está relacionada con la solución elasto-plástica de un cilindro hueco no-homogéneo, anisotrópico. Una variación potencial se ha supuesto para todas las constantes de los materiales en la etapa elástica, y también para los parámetros característicos del estado anisotrópico en la fluencia plástica. Se ha observado que la fluencia plástica puede comenzar en la superficie interior o exterior del cilindro , y existe la posibilidad de formar más de una zona plástica, dependiendo del valor de la constante del material en particular tal cual se ha visto en el caso de la barita. En forma de tabla se presentan resultados numéricos que demuestran la distribución de los esfuerzos y de las deformaciones y cómo comienza la fluencia plástica en el cilindro de boro/epoxy.

1. INTRODUCTION

Bieniek et al [1] and Maiti [2] evaluated the stresses and deformations in thickwalled orthotropic non-homogeneous elastic cylinder, while Olszak and Urbanowski [3] discussed effect of anisotropy and non-homogeneity on yield condition in state of plane stress and plane strain. As for an example, they took a body of polar orthotropy and axially symmetrical non-homogeneity, subjected to an axially symmetrical load.

In the present investigation, however, a body of cylindrical orthotropy with plane strain has been considered. Power law variation has been assumed for all material constant in elastic state and also for parameters characteristic of the state of anisotropy in plastic yield criterion.

2. FORMULATION OF THE PROBLEM

A thickwalled orthotropic cylinder in plane strain subjected to internal pressure is considered. The inner and outer radii being a and b and the z axis coincides with the axis of the cylinder. The principal directions of stresses and strains are radial, circumferencial and axial. In present case, stress-strain relations are:

$$
\begin{aligned}
\varepsilon_r &= a_{11}\sigma_r + a_{12}\sigma_\theta + a_{13}\sigma_z \\
\varepsilon_\theta &= a_{12}\sigma_r + a_{22}\sigma_\theta + a_{23}\sigma_z \qquad (2.1) \\
\varepsilon_z &= a_{13}\sigma_r + a_{23}\sigma_\theta + a_{33}\sigma_z = 0
\end{aligned}
$$

From the last relation

$$\sigma_z = -\frac{1}{a_{33}}(a_{13}\sigma_r + a_{23}\sigma_\theta) \qquad (2.2)$$

where $a_{11} = 1/E_r$, $a_{12} = -\nu_{\theta r}/E_\theta$, $a_{13} = -\nu_{zr}/E_z$, $a_{22} = 1/E_\theta$,

$$a_{23} = -\nu_{\theta z}/E_z \; , \; a_{33} = 1/E_z$$

By using relation (2.2) in (2.1) one obtains,

$$\varepsilon_r = \alpha_{11}\sigma_r + \alpha_{12}\sigma_\theta \quad , \quad \varepsilon_\theta = \alpha_{12}\sigma_r + \alpha_{22}\sigma_\theta \tag{2.3}$$

where

$$\alpha_{11} = a_{11} - \frac{a_{13}^2}{a_{33}}$$

$$\alpha_{12} = a_{12} - \frac{a_{13}\,a_{23}}{a_{33}} \tag{2.4}$$

$$\alpha_{22} = a_{22} - \frac{a_{23}^2}{a_{33}}$$

The only non vanishing equation of equilibrium (in absence of body forces) is

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{2.5}$$

The strain-displacement relations are

$$\varepsilon_r = \frac{du}{dr} \;,\; \varepsilon_\theta = \frac{u}{r} \qquad \text{where } u = u(r) \tag{2.6}$$

and hence strain compatibility condition is

$$(\varepsilon_r - \varepsilon_\theta) - r\,\frac{d\varepsilon_\theta}{dr} = 0 \tag{2.7}$$

3. POWER LAW VARIATION OF ELASTIC MATERIAL CONSTANTS

A stress function $f(r)$ is defined by

$$\sigma_r = \frac{1}{r}\frac{\partial f(r)}{\partial r} \quad \text{and} \quad \sigma_\theta = \frac{\partial^2 f(r)}{\partial r^2} \tag{3.1}$$

satisfies the equilibrium equation.
Assuming coefficients in (2.3) as

$$\alpha_{ij} = R^2\,\overline{\alpha_{ij}} \;, \quad i = 1,2 \text{ and } j = 1,2 \qquad \text{where } R = \frac{r}{a} \tag{3.2}$$

so that $\quad \dfrac{d}{dr}\,\overline{\alpha_{ij}} = 2\dfrac{R}{a}\,\overline{\alpha_{ij}}$

Considering condition (2.7) together with (2.3), (3.1) and (3.2) following differential equation is obtained

$$R^3 f''' + 3R^2 f'' + R(2\mu - \eta)\, f' = 0 \tag{3.3}$$

where prime denotes differentiation with respect to r.
Solution of (3.3) is

$$f' = D_1 R^{n_1} + D_2 R^{-n_1} \tag{3.4}$$

where $n_1 = (1+\eta-2\mu)$

Constants D_1 and D_2 are to be determined from boundary conditions

$$\sigma_r = -p \quad \text{on} \quad r = a \text{ i.e., on } R = 1$$
$$\sigma_r = 0 \quad \text{on} \quad r = b \text{ i.e., on } R = b_1 \tag{3.5}$$

where $b_1 = b/a$.

Thus from (3.1), (3.4) and (3.5) stresses in elastic state of deformations is given by

$$\frac{-\sigma_r}{p} = \frac{R^{n_1-1}}{\left(1 - b_1^{2n_1}\right)} + \frac{R^{-(n_1+1)}}{\left(1 - b_1^{-2n_1}\right)} \tag{3.6 a}$$

$$\frac{\sigma_\theta}{p} = -\frac{n_1 R^{n_1-1}}{\left(1 - b_1^{2n_1}\right)} + \frac{n_1 R^{-(n_1+1)}}{\left(1 - b_1^{-2n_1}\right)} \tag{3.6 b}$$

σ_z/p can be obtained from (2.2) with the help of (3.6 a) and (3.6 b) as

$$\frac{\sigma_z}{p} = -\xi \frac{R^{n_1+1}}{\left(1 - b_1^{2n_1}\right)} + \zeta \frac{R^{-(n_1-1)}}{\left(1 - b_1^{-2n_1}\right)} \tag{3.6 c}$$

where $\xi = \overline{\alpha_{13}} + n_1 \overline{\alpha_{23}}$, $\zeta = \overline{-\alpha_{13}} + n_1 \overline{\alpha_{23}}$

and $\quad -\dfrac{a_{13}}{a_{33}} = \alpha_{13} = R^2\,\overline{\alpha_{13}}$ and $-\dfrac{a_{23}}{a_{33}} = \alpha_{23} = R^2\,\overline{\alpha_{23}}$

4. ON SET OF PLASTIC YIELDING

In present case when non-homogeneous orthotropic cylinder in plane strain, yield condition [3,4] reduces to

$$\frac{F+G}{FG+GH+HF}\,(\sigma_r - \sigma_\theta)^2 = K^2(R) \tag{4.1}$$

where F, G, H are functions of R and defined as

$$F+G = K_1 R^{2m}\ ,\quad G+H = K_2 R^{2m}\ ,\quad H+F = K_3 R^{2m}$$

where K_1 , K_2 , K_3 are constants.
With these values, yield condition reduces to

$$(\sigma_r - \sigma_\theta)^2 = K_0^2\, R^{-2m} \tag{4.2}$$

when m is a positive integer and

$$K_0^2 = \frac{4K_1}{2K_3\left(K_1 + K_2 - \frac{1}{2}K_3\right) - (K_1 - K_2)^2} = Constant$$

From (3.6 a) and (3.6 b) it is obtained that

$$\frac{(\sigma_r - \sigma_\theta)^2}{p^2 R^{-2m}} = B^2 \left(1+AR^{-2n_1}\right)^2 R^{2(n_1+m-1)} = B^2 \Phi(R) \quad (say) \tag{4.3}$$

where
$$A = \left(\frac{1+n_1}{1-n_1}\right)\left(\frac{1-b_1^{2n_1}}{1-b_1^{-2n_1}}\right)$$

and
$$B = \frac{1-n_1}{1-b_1^{2n_1}}$$

If $\Phi(R)$ has a stationary value at $R = R_{10}$, it can be shown that

$$R_{10}^{2n_1} = A\,\frac{n_1-m+1}{n_1+m-1} \tag{4.4}$$

If $\Phi(R)$ has a minimum at $R = R_{10}$ and if it is greatest at the inner boundary it implies that plastic yielding starts at the inner surface and spreads outwards but at a certain critical value of $R = R_{20} < R_{10}$ (*say*) a second plastic zone starts at the outer surface and spreads inwards. The cylinder becomes fully plastic when these two zones meet at $R = R_{10}$.

5. ELASTO-PLASTIC SOLUTION

(a) Single plastic zone:

Let yielding start at $r=a$ and let at a particular instant, $r=c$ be the radius of plastic zone. Stresses in the elastic region $c \leq r \leq b$, are given by

$$\sigma_r = L\left\{1 + \frac{1 - b_1^{2n_1}}{1 - b_1^{-2n_1}} R^{-2n_1}\right\} R^{n_1-1}$$

$$\sigma_\theta = L n_1 \left\{1 - \frac{1 - b_1^{2n_1}}{1 - b_1^{-2n_1}}\right\} R^{n_1-1}$$

$$\sigma_z = L\left\{-(1+n_1)\,\xi - (1-n_1)\,\zeta\,\frac{1 - b_1^{2n_1}}{1 - b_1^{-2n_1}} R^{-2n_1}\right\} R^{n_1+1} \tag{5.1}$$

Considering that the material on the elastic side of $r = c$ is on the point of yielding, constant L is determined with the help of yield condition (4.2) as

$$L = K_0 \frac{C_1^{1-m-n_1}}{(1-n_1) + (1+n_1)\left(\dfrac{1 - b_1^{2n_1}}{1 - b_1^{-2n_1}}\right) C_1^{-2n_1}} \tag{5.2}$$

where $C_1 = c/a$

Stresses σ_r and σ_θ in plastic zone $a \leqslant r \leqslant c$, are obtained from the condition of equilibrium (2.5) with the help of (4.2)

$$\sigma_r = -\frac{K_0}{m} R^{-m} + B_1 \qquad (m \neq 0) \tag{5.3 a}$$

$$\sigma_\theta = \sigma_r - K_0 R^{-m} \tag{5.3 b}$$

Setting $d\theta_z = 0$ in the relations between stress and strain increment for anisotropic medium [4] σ_z is obtained as

$$\sigma_z = B_2 \sigma_r + B_3 \sigma_\theta \tag{5.3 c}$$

where $$B_2 = \frac{K_1 + (K_2 - K_3)}{2K_1}$$

$$B_3 = \frac{K_1 - (K_2 - K_3)}{2K_1}$$

Since σ_r is continuous across plastic boundary $r = c$, constant B_1 in (5.3 a) is obtained with the help of (5.1) and (5.2) as

$$B_1 = \frac{K_0 C_1^{-m}}{m}\left\{1 + \frac{m\left[\left(1 - b_1^{-2n_1}\right) + \left(1 - b_1^{2n_1}\right) C_1^{-2n_1}\right]}{\left[\left(1 - b_1^{-2n_1}\right)(1-n_1) + \left(1 - b_1^{2n_1}\right)(1+n_1)\, C_1^{-2n_1}\right]}\right\} \tag{5.4}$$

(b) Double plastic zones:

In case of possibility of double plastic zones, let $r = c$ or $R = C_1$ be the boundary of the inner plastic zone when yield starts at $r = b$ or $R = b_1$. Considering yield criterion at the outer boundary value of C_1 can be obtained from:

$$\frac{4n_1^2\left(1-b_1^{-2n_1}\right)^2 (b_1/c_1)^{2n_1+2m-2}}{\left\{(1-n_1)\left(1-b_1^{-2n_1}\right) + (1+n_1)\left(1-b_1^{2n_1}\right) C_1^{-2n_1}\right\}^2} \tag{5.5}$$

Let p_2 be the pressure at which the inner plastic zone and outer plastic zone has radius $r = d(>c)$ and $r = e(<b)$ respectively. σ_r in the plastic zones can be obtained from the equation of equilibrium with the help of yield condition as

$$\sigma_r = -\frac{K_0}{m}\left(\frac{r}{a}\right)^{-m} + A \tag{5.6}$$

In the plastic zone $a \leqslant r \leqslant d$, boundary condition $\sigma_r = -p_2$ at $r = a$ gives

$$\sigma_r = -\frac{K_0}{m} R^{-m} + \frac{K_0}{m} - p_2 \qquad (5.7\,a)$$

$$\sigma_\theta = \sigma_r - K_0 R^{-m} = -K_0 R^{-m}\left(1 + \frac{1}{m}\right) + \frac{K_0}{m} - p_2 \qquad (5.7\,b)$$

$$\sigma_z = -\frac{K_0}{m}(B_2+B_3) R^{-m} - B_3 K_0 R^{-m} + \left(\frac{K_0}{m} - p_2\right)(B_2+B_3) \qquad (5.7\,c)$$

In the plastic zone $e \leqslant r \leqslant b$, boundary condition $\sigma_r = 0$ at $r = b$ gives:

$$\sigma_r = -\frac{K_0}{m}\left(R^{-m} - b_1^{-m}\right) \qquad (5.8\,a)$$

$$\sigma_\theta = -K_0 R^{-m}\left(1 + \frac{1}{m}\right) + \frac{K_0}{m} b_1^{-m} \qquad (5.8\,b)$$

$$\sigma_z = -\frac{K_0}{m}(B_2+B_3) R^{-m} - B_3 K_0 R^{-m} + \frac{K_0}{m}(B_2+B_3) b_1^{-m} \qquad (5.8\,c)$$

From the value of f' in (3.4) and from the condition of continuity across the boundaries $r = d$ and $r = e$, stresses in the elastic zone $d \leqslant r \leqslant e$, are given by:

$$\sigma_r = -\frac{1}{\lambda}\left[I_1 e_1^{-(n_1+1)} - I_2 d_1^{-(n_1+1)}\right] R^{n_1 - 1}$$

$$+ \frac{1}{\lambda}\left[I_1 e_1^{(n_1-1)} - I_2 d_1^{(n_1-1)}\right] R^{-(n_1+1)} \tag{5.9 a}$$

$$\sigma_\theta = -\frac{n_1}{\lambda}\left[I_1 e_1^{-(n_1+1)} - I_2 d_1^{-(n_1+1)}\right] R^{n_1-1}$$

$$-\frac{n_1}{\lambda}\left[I_1 e_1^{(n_1-1)} - I_2 d_1^{(n_1-1)}\right] R^{-(n_1+1)} \tag{5.9 b}$$

$$\sigma_z = -\frac{\xi}{\lambda}\left[I_1 e_1^{-(n_1+1)} - I_2 d_1^{-(n_1+1)}\right] R^{n_1+1}$$

$$-\frac{\xi}{\lambda}\left[I_1 e_1^{(n_1-1)} - I_2 d_1^{(n_1-1)}\right] R^{-(n_1-1)} \tag{5.9 c}$$

where $$I_1 = -\frac{K_0}{m}(d_1^{-m} - 1) - p_2$$

$$I_2 = -\frac{K_0}{m}(e_1^{-m} - b_1^{-m})$$

$$\lambda = (d_1 e_1)^{n_1-1}\left[d_1^{-2n_1} - e_1^{-2n_1}\right]; \quad \frac{d}{a} = d_1, \quad \frac{e}{a} = e_1$$

Since the yield condition (4.2) holds at boundaries $r = d$ and $r = e$, unknown pressure p_2 and radius e_1 will be determined as a function of d_1 from those two conditions, where values of σ_r and σ_θ will be taken from region $d \leqslant r \leqslant e$.

Numerical Examples:

(a) Material - Baryte[5]

$$\frac{1}{E_r} = 1.84 \times 10^{-12} cm^2/dyne \; ; \; \frac{1}{E_\theta} = 1.74 \times 10^{-12} cm^2/dyne$$

$$\frac{1}{E_z} = 1.10 \times 10^{-12} cm^2/dyne \; ; \; \frac{\nu_{\theta r}}{E_\theta} = 0.95 \times 10^{-12} cm^2/dyne$$

$$\frac{\nu_{z\theta}}{E_z} = \frac{\nu_{zr}}{E_z} = (0.27 \times 10^{-12})\ cm^2/dyne$$

From table I it is clear that for $m = 2.5$, plastic flow starts at the outer surface and spreads inwards, but at a certain critical value $R = R_{20} > R_{10}$, a second plastic zone starts at the inner surface. The cylinder will be fully plastic when these two zones meet at $R_{10} = 1.462$.

(b) Material - Boron/epoxy[2]

$$E_\theta = E_z = 3.6 \times 10^6\ Lb/in^2 \; , \; E_r = 30.8 \times 10^6\ Lb/in^2$$

$$\nu_{zr} = \nu_{\theta r} = 0.0421 \; , \; \nu_{\theta z} = 0.3$$

Table II shows that there is only one plastic zone and for $m \leqslant 2$, it starts at the inner boundary while for $m > 2$, plastic zone starts at the outer boundary.

Numerical values of σ_r/K_0 and σ_θ/K_0 in elastic and plastic zones in a boron/epoxy cylinder are given in tabular form. It is assumed that the range $1.0 \leqslant R \leqslant 1.5$ is plastic and the range $1.5 \leqslant R \leqslant 2.0$ is elas -

tic. Tables III and IV and Tables V and VI show that an increase in the degree of anisotropy reduces the magnitude of the stresses and elastic strains respectively. Table VII shows that plastic strain increment $d\varepsilon_r/d\lambda$ either remains constant or decreases.

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Table I : $\left[\text{values of } (\sigma_r - \sigma_\theta)^2/(p^2 R^{-2m}) \;;\; b_1 = 2.0\right]$

m \ R	1.0	1.2	1.4	1.6	1.8	2.0
0.5	9.840	4.426	2.317	1.369	0.8923	0.6403
1.0	9.840	5.313	3.245	2.190	1.606	1.276
2.0	9.840	7.649	6.246	5.610	5.202	5.126
2.5	9.840	9.179	8.902	8.974	9.380	10.250
3.0	9.840	11.020	12.470	14.320	16.880	20.500

Table II : $\left[\text{values of } \dfrac{(\sigma_r - \sigma_\theta)^2}{p^2 R^{-2m}} \;;\; b = 2.0\right]$

m \ R	1.0	1.2	1.4	1.6	1.8	2.0
1.0	7.396	4.997	3.579	2.704	2.118	1.711
1.5	7.396	5.973	5.010	4.326	3.813	3.421
2.0	7.396	7.168	7.013	6.918	6.858	6.845
2.5	7.396	8.602	9.819	11.080	12.370	13.690
3.0	7.396	10.320	13.740	17.720	22.240	27.380

Table III : [Values of σ_r / k_o , $c_1 = 1.5$, $b_1 = 2.0$]

	Plastic zone					Elastic zone				
n \ R	1.0	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
0.5	-0.18281	-0.00855	0.06307	0.12688	0.18420	0.12832	0.08753	0.05342	0.02459	0
1.0	-0.18294	-0.01627	0.04783	0.10278	0.15040	0.10477	0.07147	0.04356	0.02008	0
1.5	-0.18098	-0.02147	0.03591	0.08323	0.12280	0.08555	0.05835	0.03561	0.01639	0
2.0	-0.17752	-0.02474	0.02663	0.06739	0.10026	0.06985	0.04765	0.02908	0.01338	0

Table IV : [Values of $-\sigma_\theta / k_o$, $c_1 = 1.5$, $b_1 = 2.0$]

	Plastic zone					Elastic zone				
n \ R	1.0	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
0.5	1.18281	0.92142	0.81399	0.71827	0.63230	0.58699	0.54459	0.50948	0.48007	0.45537
1.0	1.18294	0.84960	0.7[illegible]140	0.61151	0.51627	0.47928	0.44465	0.41599	0.39202	0.37131
1.5	1.18098	0.78219	0.63875	0.52045	0.42153	0.39133	0.35306	0.33945	0.32008	0.3035[illegible]
2.0	1.17758	0.71918	0.56509	0.44282	0.34418	0.31952	0.22644	0.27732	0.26135	0.24787

Table V : Values of $\epsilon_r \times 10^3$ in Elastic zone $(c_1 : 1.5 , b_1 : 2.0)$

m \ R	1.5	1.6	1.7	1.8	1.9	2.0
0.5	15.46937	13.08278	11.1160[illegible]	9.47372	8.0[illegible]523	6.42082
1.0	12.63110	10.68205	8.99153	7.72[illegible]	6.61913	5.6[illegible]
1.5	10.31324	8.72185	7.41070	6.[illegible]1470	5.39739	4.61522
2.0	8.40438	7.12136	6.05081	5.16470	4.40693	3.76814

Table VI : Values of $-\epsilon_\theta \times 10^5$ in Elastic zone : $(c_1 = 1.5 , b_1 = 2.0)$

m \ R	1.5	1.6	1.7	1.8	1.9	2.0
0.5	0.[illegible]61789	0.148607	0.1378[illegible]4	0.1[illegible]8879	0.121394	0.1151076
1.0	0.132110	0.121337	0.112525	0.105229	0.099130	0.093985
1.5	0.107859	0.099071	0.09[illegible]876	0.085668	0.080939	0.076738
2.0	0.0880664	0.080391	0.075017	0.070254	0.066086	0.062657

Table VII : Values of Plastic strain increment $d\epsilon_r / d\lambda = (\sigma_r - \sigma_\theta)$ [4]

m \ k	1.0	1.2	1.3	1.4	1.5
0.5	1.0	0.91287	0.[illegible]	0.84516	0.81750
1.0	1.0	0.83333	0.76923	0.71429	0.66667
1.5	1.0	0.75972	0.[illegible]7460	0.60368	0.54433
2.0	1.0	0.69444	0.[illegible]172	0.51020	0.44444

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