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HEAT CONDUCTION IN A TRUNCATED WEDGE OF SEMI-INFINITE HEIGHT WITH CIRCULAR BOUNDARIES

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ABSTRACT

In this paper, the problem of finding the temperature at any point of a truncated wedge of semi-infinite height, with circular boundaries, when there are sources of heat within it, for boundary conditions of the radiation type is considered. Different types of integral transforms are employed to solve the problem.

RESUMEN

En este artículo se considera el problema de evaluar la temperatura en cualquier punto de una cuña truncada de altura semi-infinita y bordes circulares. Existen fuentes de energía dentro del sistema y las condiciones de contorno son de tipo radiación. Varios tipos de transformadas de Laplace son usadas para resolver el problema.

1. INTRODUCTION

Marchi and Zgrablich [5] have solved the problem of finding the temperature at any point of a hollow cylinder of any height with heat radiation on its surfaces using extended finite Hankel transform and sine transform. The object of this paper is to generalize this problem by considering an angular wedge section of a semi-infinite hollow cylinder with internal heat generation and radiation type time dependent boundary conditions.

2. FORMULATION OF THE PROBLEM

For the truncated wedge shown in Fig.1, the governing differential equation [1,2] for conduction of heat is

$$\frac{\partial u}{\partial t} = \alpha \left\{ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} \right\} + \phi(r,\theta,z,t) \tag{1}$$

$$z > 0, \quad a < r < b, \quad 0 < \theta < \delta$$

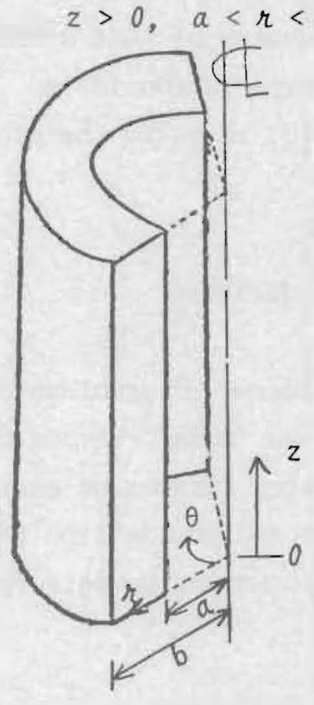

Fig.1 Physical Model and Coordinates

The initial and boundary conditions considered are

$$u = E(r,\theta,z) \quad ; \quad t = 0 \quad 0 < \theta < \delta \quad a < r < b \quad z > 0 \tag{2}$$

$$k_1 \frac{\partial u}{\partial r} + u = F(\theta,z,t) \quad ; \quad t > 0 \quad 0 < \theta < \delta \quad r = a \quad z > 0 \tag{3}$$

$$k_2 \frac{\partial u}{\partial r} + u = H(\theta,z,t) \quad ; \quad t > 0 \quad 0 < \theta < \delta \quad r = b \quad z > 0 \tag{4}$$

$$u = G(r,\theta,t) \quad ; \quad t > 0 \quad 0 < \theta < \delta \quad a < r < b \quad z = 0 \tag{5}$$

$$u = \text{finite} \quad ; \quad t > 0 \quad 0 < \theta < \delta \quad a < r < b \quad z \to \infty \tag{6}$$

$$u = W(r,z,t) \quad ; \quad t > 0 \quad \theta = 0 \quad a < r < b \quad z > 0 \tag{7}$$

$$u = Z(r,z,t) \quad ; \quad t > 0 \quad \theta = \delta \quad a < r < b \quad z > 0 \tag{8}$$

3. THE SOLUTION OF THE PROBLEM

Multiplying both sides of equation (1) by $\sin \frac{m_1 \pi \theta}{\delta} d\theta$, integrating from 0 to δ and using the property (A.3) of finite sine transform and the boundary conditions (7), and (8) we obtain the finite sine transform w.r.t. θ of equation (1).

$$\frac{\partial u_s}{\partial t} = \alpha \left\{ \frac{\partial^2 u_s}{\partial r^2} + \frac{1}{r}\frac{\partial u_s}{\partial r} + \frac{m_1 \pi}{r^2 \delta}\left[(-1)^{m_1+1} z(r,z,t) + W(r,z,t)\right] \right.$$

$$\left. - \frac{m_1^2 \pi^2}{r^2 \delta^2} u_s + \frac{\partial^2 u_s}{\partial z^2} \right\} + \phi_s \tag{9}$$

where $u_s = u_s(r,m_1,z,t) = \int_0^\delta u(r,\theta,z,t) \sin \frac{m_1 \pi \theta}{\delta} d\theta$

$$\phi_s = \phi_s(r,m_1,z,t) = \int_0^{\delta} \phi(r,\theta,z,t)\, \sin \frac{m_1 \pi \theta}{\delta}\, d\theta .$$

Let $p = \dfrac{m_1 \pi}{\delta}$ and $\psi(r,m_1,z,t) = \left[(-1)^{m_1+1} Z(r,z,t) + W(r,z,t)\right] \dfrac{1}{r^2}$

Thus equation (9) may be written as

$$\frac{\partial u_s}{\partial t} = \alpha \left\{ \frac{\partial^2 u_s}{\partial r^2} + \frac{1}{r} \frac{\partial u_s}{\partial r} - \frac{p^2}{r^2} u_s + p\psi + \frac{\partial^2 u_s}{\partial z^2} \right\} + \phi_s(r,m_1,z,t) \tag{10}$$

Now multiplying both sides of equation (10) by the Kernel $r\, s_p(k_1,k_2,\mu_n r)$, integrating from a to b, and using the generalized Hankel transform property (A.11) we get the following equation.

$$\frac{\partial \bar{u}_s}{\partial t} = \alpha \left\{ \frac{b}{k^2}\, s_p(k_1,k_2,\mu_n b) \left[u_s + k_2 \frac{\partial u_s}{\partial r} \right]_{r=b} - \frac{a}{k_1}\, s_p(k_1,k_2,\mu_n a) \right.$$

$$\left. \times \left[u_s + k_1 \frac{\partial u_s}{\partial r} \right]_{r=a} - \mu_n^2 \bar{u}_s + p\bar{\psi} + \frac{\partial^2 u_s}{\partial z^2} \right\} + \bar{\phi}_s \tag{11}$$

where:

The function $s_p(k_1,k_2,\mu_n r)$ is defined as in equation (A.5)

$$\bar{u}_s = \bar{u}_s(n,m_1,z,t) = \int_a^b r\, u_s(r,m_1,z,t)\; s_p(k_1,k_2,\mu_n r)\, dr$$

$$\bar{\phi}_s = \bar{\phi}_s(n,m_1,z,t) = \int_a^b r\, \phi_s(r,m_1,z,t)\; s_p(k_1,k_2,\mu_n r)\, dr$$

$$\bar{\psi} = \bar{\psi}(n,m_1,z,t) = \int_a^b r\, \psi(r,m_1,z,t)\; s_p(k_1,k_2,\mu_n r)\, dr$$

boundary conditions

The finite sine transform of the (3) and (4) gives the following

$$\left[k_1 \frac{\partial u_s}{\partial r} + u_s\right]_{r=a} = F_s(m_1,z,t) \qquad (12)$$

and

$$\left[k_2 \frac{\partial u_s}{\partial r} + u_s\right]_{r=b} = H_s(m_1,z,t) \qquad (13)$$

Substituting (12) and (13) in equation (11) we get

$$\frac{\partial \bar{u}_s}{\partial t} = \alpha \left\{ X - \mu_n^2 \bar{u}_s + p\bar{\psi} + \frac{\partial^2 \bar{u}_s}{\partial z^2} \right\} + \bar{\phi}_s \qquad (14)$$

where $X = X(n,m_1,z,t)$

$$= \frac{b}{k_2} s_p(k_1,k_2,\mu_n b)\, H_s(m_1,z,t) - \frac{a}{k_1} s_p(k_1,k_2,\mu_n a)\, F_s(m_1,z,t) \tag{15}$$

Now multiplying both sides of equation (14) by $\sqrt{\frac{2}{\pi}}\, sin\, m_2 z$, integrating from 0 to ∞, and using the property (A.14) we obtain the following Fourier sine transform of equation (14)

$$\frac{\bar{u}_{s,s_1}}{\partial t} = \alpha \left\{ X_{s_1} - \mu_n^2 \bar{u}_{s,s_1} - m_2^2 \bar{u}_{s,s_1} + m_2\, \bar{u}_s(n,m_1,0,t) + p\bar{\psi}_{s_1} \right\} + \bar{\phi}_{s,s_1} \tag{16}$$

where

$$\bar{u}_{s,s_1} = \bar{u}_{s,s_1}(n,m_1,m_2,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{u}_s(n,m_1,z,t)\, sin\, m_2 z\, dz \tag{17}$$

$$X_{s_1} = X_{s_1}(n,m_1,m_2,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty X(n,m_1,z,t)\, sin\, m_2 z\, dz \tag{18}$$

$$\bar{\psi}_{s_1} = \bar{\psi}_{s_1}(n,m_1,m_2,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{\psi}(n,m_1,z,t)\, sin\, m_2 z\, dz \tag{19}$$

$$\overline{\phi}_{s,s_1} = \overline{\phi}_{s,s_1}(n,m_1,m_2,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \overline{\phi}_s(n,m_1,z,t)\, \sin m_2 z \, dz \qquad (20)$$

from boundary condition (5)

$$u_s(r,m_1,0,t) = \int_0^\delta G(r,\theta,t) \, \sin \frac{m_1 \pi \theta}{\delta} \, d\theta = G_s(r,m_1,t) \qquad (21)$$

$$\overline{u}_s(n,m_1,0,t) = \int_a^b r\, G_s(r,m_1,t)\, s_p(k_1,k_2,\mu_n r)\, dr = \overline{G}_s(n,m_1,t) \qquad (22)$$

so equation (16) can be written as

$$\frac{d\overline{u}_{s,s_1}}{dt} = \alpha \left\{ X_{s_1} - \mu_n^2 \overline{u}_{s,s_1} - m_2^2 \overline{u}_{s,s_1} + m_2 \overline{G}_s + p \overline{\psi}_{s_1} \right\} + \overline{\phi}_{s,s_1} \qquad (23)$$

Now taking Laplace transform w.r.t. time of eqn. (23) we get

$$qL\left[\overline{u}_{s,s_1}\right] - \overline{u}_{s,s_1}(n,m_1,m_2,0) = \alpha L\left[X_{s_1}\right] - \alpha \mu_n^2 L\left[\overline{u}_{s,s_1}\right]$$

$$- \alpha m_2^2 L\left[\overline{u}_{s,s_1}\right] + \alpha m_2 L\left[\overline{G}_s\right] + \alpha p L\left[\overline{\psi}_{s_1}\right] + L\left[\overline{\phi}_{s,s_1}\right] \qquad (24)$$

where $$L\left[\bar{u}_{s,s_1}\right] = \int_0^\infty e^{-qt}\,\bar{u}_{s,s_1}\,dt \quad \text{etc.}$$

From b c. (2)

$$\bar{u}_{s,s_1}(n,m_1,m_2,0) = \sqrt{\frac{2}{\pi}} \int_0^\infty \int_a^b \int_0^\delta r\,E(r,\theta,z)\,\sin\frac{m_1\pi\theta}{\delta}$$

$$\times\; s_p(k_1,k_2,\mu_n r)\,\sin m_2 z\;d\theta\,dr\,dz = \bar{E}_{s,s_1}(n,m_1,m_2) \qquad (25)$$

Thus equation (24) can be written as

$$(q+A)\,L\left[\bar{u}_{s,s_1}\right] = \alpha L\left[X_{s_1}\right] + \alpha m_2 L\left[\bar{G}_s\right] + \alpha p L\left[\bar{\psi}_{s_1}\right]$$

$$+\, L\left[\bar{\phi}_{s,s_1}\right] + \bar{E}_{s,s_1} \qquad (26)$$

where $$A = \alpha(\mu_n^2 + m_2^2)$$

equation (26) may be written as

$$L\left[\bar{u}_{s,s_1}\right] = \frac{\alpha}{q+A}\,L\left[X_{s_1}\right] + \frac{\alpha m_2}{q+A}\,L\left[\bar{G}_s\right] + \frac{\alpha p}{q+A}\,L\left[\bar{\psi}_{s_1}\right]$$

$$+\,\frac{1}{q+A}\,L\left[\bar{\phi}_{s,s_1}\right] + \frac{\bar{E}_{s,s_1}}{q+A} \qquad (27)$$

Now taking the inverse Laplace transform of the above equation and using the convolution property we get

$$\overline{u}_{s,s_1} = \alpha \int_0^t e^{-A(t-T)} X_{s_1} \, dT + \alpha m_2 \int_0^t e^{-A(t-T)} \overline{G}_s \, dT$$

$$+ \alpha p \int_0^t e^{-A(t-T)} \overline{\psi}_{s_1} \, dT + \int_0^t e^{-A(t-T)} \overline{\phi}_{s,s_1} \, dT$$

$$+ e^{-At} \overline{E}_{s,s_1} \tag{28}$$

Now applying the inverse sine transform as defined by eqn. (A.13) we get

$$\overline{u}_s = \alpha\sqrt{\frac{2}{\pi}} \int_0^\infty \int_0^t e^{-A(t-T)} X_{s_1} \sin m_2 z \, dT \cdot dm_2$$

$$+ \alpha\sqrt{\frac{2}{\pi}} \int_0^\infty \int_0^t m_2 \, e^{-A(t-T)} \overline{G}_s \sin m_2 z \, dT \cdot dm_2$$

$$+ \alpha p\sqrt{\frac{2}{\pi}} \int_0^\infty \int_0^t e^{-A(t-T)} \overline{\psi}_{s_1} \sin m_2 z \, dT \, dm_2$$

$$+\sqrt{\frac{2}{\pi}}\int_0^\infty\int_0^t e^{-A(t-T)}\,\overline{\phi}_{s,s_1}\,\sin m_2 z\;dT\cdot dm_2$$

$$+\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-At}\,\overline{E}_{s,s_1}\;dm_2\;. \qquad (29)$$

Now applying the inverse generalized Hankel transform as defined by eqn. (A.9) we get

$$u_s=\sqrt{\frac{2}{\pi}}\sum_n^\infty \frac{1}{c_n}\,s_p(k_1,k_2,\mu_n r)\left[\alpha\int_0^\infty\int_0^t e^{-A(t-T)}\,X_{s_1}\,\sin m_2 z\;dT\;dm_2\right.$$

$$+\alpha\int_0^\infty\int_0^t m_2 e^{-A(t-T)}\,\overline{G}_s\,\sin m_2 z\;dT\cdot dm_2$$

$$+\alpha p\int_0^\infty\int_0^t e^{-A(t-T)}\,\overline{\psi}_{s_1}\,\sin m_2 z\;dT\;dm_2$$

$$\left.+\int_0^\infty\int_0^t e^{-A(t-T)}\,\overline{\phi}_{s,s_1}\,\sin m_2 z\;dT\cdot dm_2+\int_0^\infty e^{-At}\,\overline{E}_{s,s_1}\;dm_2\right] \qquad (30)$$

Now finally applying the inverse finite sine transform as defined by (A.2) we get the required temperature distribution function $u(r,\theta,z,t)$.

$$u(r,\theta,z,t) = \sqrt{\frac{2}{\pi}} \cdot \frac{2}{\delta} \sum_{m_1=1}^{\infty} \sum_{n}^{\infty} \frac{1}{c_n} s_p(k_1,k_2,\mu_n r)$$

$$\times \sin \frac{m_1 \pi\theta}{\delta} \left[\alpha \int_0^{\infty} \int_0^{t} e^{-A(t-T)} X_{s_1} \sin m_2 z \; dT \; dm_2 \right.$$

$$+ \alpha \int_0^{\infty} \int_0^{t} m_2 e^{-A(t-T)} \overline{G}_s \sin m_2 z \; dT \cdot dm_2$$

$$+ \alpha p \int_0^{\infty} \int_0^{t} e^{-A(t-T)} \overline{\psi}_{s_1} \sin m_2 z \; dT \; dm_2$$

$$+ \int_0^{\infty} \int_0^{t} e^{-A(t-T)} \overline{\phi}_{s,s_1} \sin m_2 z \; dT \cdot dm_2$$

$$+ \int_0^{\infty} e^{-At} \overline{E}_{s,s_1} \; dm_2 \qquad (31)$$

4. ESPECIAL CASE

Taking u and ϕ as independent of θ in (1) and adjusting the boundary conditions accordingly, we have the following conduction equation

$$\frac{\partial u}{\partial t} = \alpha \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} + \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} \right) + \phi(r,z,t) \qquad (32)$$

$$z > 0 \quad , \quad a < r < b$$

with initial and boundary conditions

$$u = E(r,z) \;;\; t = 0 \;,\; a < r < b \;,\; z > 0 \qquad (33)$$

$$k_1 \frac{\partial u}{\partial r} + u = F(z,t) \;;\; r = a \;,\; z > 0 \;,\; t > 0 \qquad (34)$$

$$k_2 \frac{\partial u}{\partial r} + u = H(z,t) \;;\; r = b \;,\; z > 0 \;,\; t > 0 \qquad (35)$$

$$u = G(r,t) \;;\; z = 0 \;,\; a < r < b \;,\; t > 0 \qquad (36)$$

$$u = \mathit{finite} \;;\; z \to \infty \;,\; a < r < b \;,\; t > 0 \qquad (37)$$

On applying the generalized Hankel transform (A.4) with $p = 0$ and Fourier sine transform (A.12) in succession to equation (32) and the boundary conditions (33) to (37), the temperature at any point of a semi-infinite hollow cylinder defined by $z > 0$, $a < r < b$ having symmetry about the axis of the cylinder and sources of heat within it and with heat radiation on its surfaces is given by

$$u(r,z,t) = \sqrt{\frac{2}{\pi}} \sum_n \frac{1}{c_n} s_0(\mu_n r) \int_0^\infty \bar{u}_{s_1}(n,m_2,t) \sin m_2 z \, dm_2 \qquad (38)$$

where

$$\bar{u}_{s_1}(n,m_2,t) = \bar{E}_{s_1}(n,m_2)\, e^{-At} + \alpha \int_0^t Y_{s_1}(m_2,T)\, e^{-A(t-T)}\, dT$$

$$+ \sqrt{\frac{2}{\pi}}\, m_2 \alpha \int_0^t \bar{G}(n,T)\, e^{-A(t-T)}\, dT$$

$$+ \int_0^t \bar{\phi}_{s_1}(n,m_2,T)\, e^{-A(t-T)}\, dT \qquad (39)$$

$$y_{s_1}(m_2,t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} y(z,t)\, \sin m_2 z\, dz$$

and

$$y(z,t) = \frac{b}{k_2}\, s_0(\mu_n b)\, H(z,t) - \frac{a}{k_1}\, s_0(\mu_n a)\, F(z,t)$$

EXAMPLES

First Example

Let $E(r,\theta,z) = r^{Q-1} z^{-\frac{1}{2}}$ (a function independent of θ) (40)

where Q is a real number

$$F(\theta,z,t) = z(h^2 + z^2)^{-1} e^{-Bt} \theta^{d-1} (\delta^2 - \theta^2)^{-\frac{\mu}{2}} P_\nu^\mu\left(\frac{\theta}{\delta}\right) \qquad (41)$$

where h and B are real numbers, d is a complex number with $Re(d) > -1$, P_ν^μ is an associated Legendre function [3, p.120] with $\mu < 1$.

$$H(\theta,z,t) = 0 \qquad (42)$$

$$G(r,t) = r^{Q-1} \quad \text{(a function independent of } t\text{)} \qquad (43)$$

$$W(r,z,t) = 0 \qquad (44)$$

$$Z(r,z,t) = 0 \qquad (45)$$

$$\phi(r,\theta,z,t) = r^{Q-1}\, \theta(\delta^2 - \theta^2)^{M-1} z^{N-1} \delta_1(t) \qquad (46)$$

where M is a real number, $0 < N < 1$, and $\delta_1(t)$ is Dirac delta function.

Taking the above functions in the problem represented by equations (1) to (8), we get the following value of $\bar{u}_{s,s_1}$ with the help of equations (15), (18), (22), (25), and (28), and the know results [9, p.518; 3, p.90, (7); p.46, (51)]

$$\bar{u}_{s,s_1}(n,m_1,m_2,t)$$

$$= \frac{\alpha a s_p(\mu_n a)\, m_1\, \pi^2\, \delta^{d-\mu}\, 2^{\mu-d-\frac{3}{2}}\, \Gamma(1+d)\, e^{-hm_2}}{k_1\, \Gamma\left(\frac{2+d-\mu-\nu}{2}\right)\, \Gamma\left(\frac{3+d-\mu+\nu}{2}\right)\, (A-B)} \left(e^{-At} - e^{-Bt}\right)$$

$$\times {}_2F_3\left[\begin{matrix} \frac{1+d}{2}, & \frac{2+d}{2} & \\ \frac{3}{2}, & \frac{2+d-\mu-\nu}{2}, & \frac{3+d-\mu-\nu}{2} \end{matrix} ;\ -\frac{m_1^2\,\pi^2}{4}\right]$$

$$+ R_4\, e^{-At}\left[\frac{\left\{1+(-1)^{m_1+1}\right\}\delta}{\sqrt{m_2}\,(m_1\pi)} - \frac{\alpha\beta}{A}(1-e^{At})\right.$$

$$\left. + \sqrt{2}\left(\frac{2}{m_1\pi}\right)^{M-\frac{1}{2}} \frac{\delta^{2M}}{(m_2)^N}\, \Gamma(M)\, \Gamma(N)\, \sin\left(\frac{\pi}{2}N\right) J_{M+\frac{1}{2}}(m_1\pi)\right]. \qquad (47)$$

where ${}_2F_3$ is the generalized hypergeometric function [3]

$$\beta = \sqrt{\frac{2}{\pi}}\ \frac{m_2\,\delta}{\pi m_1}\left[1-(-1)^{m_1}\right]$$

$$R_4 = R_3(\mu_n)^{-Q}\Big[b\Big\{(p+Q-1)\ J_p(\mu_n b)\ s^*_{Q-1,p-1}(\mu_n b)$$

$$-\ J_{p-1}(\mu_n b)\ s^*_{Qp}(\mu_n b)\Big\}$$

$$-\ a\Big\{(p+Q-1)\ J_p(\mu_n a)\ s^*_{Q-1,p-1}(\mu_n a)$$

$$-\ J_{p-1}(\mu_n a)\ s^*_{Qp}(\mu_n a)\Big\}\Big] \qquad (48)$$

$$R_3 = R_1 - \frac{1}{2}R_2\ \mathrm{Cosec}\ (p\pi)\left[(-1)^p - e^{-ip\pi}\right] \qquad (49)$$

$$R_1 = G_p(k_1,\mu_n a) + G_p(k_2,\mu_n b) \qquad (50)$$

$$R_2 = J_p(k_1,\mu_n a) + J_p(k_2,\mu_n b) \qquad (51)$$

$$G_p(k_i,\mu_n x) = G_p(\mu_n x) + k_i G'_p(\mu_n x) \qquad (52)$$

$$G_p(\mu_n x) = \frac{1}{2}\ \mathit{Cosec}\ p\pi \left[J_{-p}(\mu_n x) - e^{-ip\pi}\ J_p(\mu_n x)\right] \tag{53}$$

$s^{*}_{p,\ell}(\mu_n x)$ is Lommel function given by [3, vol. II, p.40 (71)]

Therefore the temperature $u(r,\theta,z,t)$ is given by equation (31) after applying inversion formulas (A.13) , (A.9) , and (A.2) in succession to equation (47). The procedure of applying inversion formulas is shown by equations (29) , (30) , and (31).

Second Example

Let $E(r,\theta,z) = r^{\ell-1}\ e^{-kz}$ (a function independent of θ) (54)

$$F(\theta,z,t) = 0 \tag{55}$$

$$H(\theta,z,t) = z\,e^{-h^2 z^2}\ \delta_1(t)\ \theta\left(1-\frac{\theta^2}{\delta^2}\right)^{\frac{\nu}{2}} J_\nu\left(\beta\sqrt{1-\left(\frac{\theta^2}{\delta^2}\right)}\right),\ \left(Re(\nu) > -1\right) \tag{56}$$

$$G(r,\theta,t) = W(r,z,t) = Z(r,z,t) = \phi(r,\theta,z,t) = 0 \tag{57}$$

Taking the above functions in the problem represented by equations (1) to (8), and evaluating the integrals involved in equation (25) with the help of known results [3, p.46 (4); 9, p.518], we get the following solution subject to the corresponding boundary conditions.

$$u(r,\theta,z,t) = \sum_{m_1=0}^{\infty} \sum_n \frac{1}{c_n} \sin \frac{m_1 \pi \theta}{\delta} s_p(\mu_n r) \left[4(m_1 \pi^2)^{-1} R_4 \left\{ 1 + (-1)^{m_1+1} \right\} \right.$$

$$\times \int_0^{\infty} m_2 (k^2 + m_2^2)^{-1} \exp\left\{ -\alpha(\mu_n^2 + m_2^2)\, t \right\} \sin m_2 z \, dm_2$$

$$+ \delta \alpha b m_1 \pi 2^{-\frac{1}{2}} h^{-3} (\beta)^{\nu} (m_1^2 \pi^2 + \beta^2)^{-\frac{\nu}{2} - \frac{3}{4}} J_{\nu+\frac{3}{2}} (m_1^2 \pi^2 + \beta^2) s_p(\mu_n b)$$

$$\left. \times \int_0^{\infty} m_2 \exp - \left\{ \left(\frac{m_2^2}{4h^2} \right) - \alpha(\mu_n^2 + m_2^2)\, t \right\} \sin m_2 z \, dm_2 \right] \qquad (58)$$

Lastly, on evaluating the m_2 integrals involved in (58) with the help of known results [4, p.73 (19) ; p.74 (26)], we get the solution of the boundary value problem in the following form.

$$u(r,\theta,z,t) = \sum_{m_1=0}^{\infty} \sum_n \frac{1}{c_n} \sin \frac{m_1 \pi \theta}{\delta} s_p(\mu_n r)\, e^{-\mu_n^2 \alpha t} \left[(m_1 \pi)^{-1} R_4 \left\{ 1 \right. \right.$$

$$\left. + (-1)^{m_1+1} \right\} e^{k^2 \alpha t} \left\{ e^{-kz} \operatorname{Erfc}\left(k\sqrt{\alpha t} - \frac{z}{2\sqrt{\alpha t}} \right) \right.$$

$$\left. - e^{kz} \operatorname{Erfc}\left(k\sqrt{\alpha t} + \frac{z}{2\sqrt{\alpha t}} \right) \right\} + \delta \alpha b m_1 2^{-\frac{5}{2}}$$

$$\times \sqrt{\pi}\, h^{-3}\, \beta^{\nu}\, (m_1^2\, \pi^2 + \beta^2)^{-\frac{\nu}{2}+\frac{3}{4}}\, J_{\nu+\frac{3}{2}}\, (m_1^2\, \pi^2 + \beta^2)$$

$$\times\, s_p(\mu_n b)\; z \left(\frac{1}{4\,h^2} + \alpha t\right)^{-\frac{3}{2}} e^{-\frac{1}{4}\left(\alpha t + \frac{1}{4h^2}\right)^{-1} z^2} \Bigg] \tag{59}$$

where the value of R_4 is given by (48).

APPENDIX

The finite sine transform [8,9] of a function $f(x)$ is defined as

$$f_s(m) = \int_0^h f(x)\, \sin \frac{m\,\pi\,x}{h}\, dx \tag{A.1}$$

where
$$f(x) = \frac{2}{h} \sum_{m=1}^{\infty} f_s(m)\, \sin \frac{m\,\pi\,x}{h} \tag{A.2}$$

It has a following property

$$\int_0^h \frac{\partial^2 f}{\partial x^2}\, \sin \frac{m\,\pi\,x}{h}\, dx = \frac{m\,\pi}{h}\left[(-1)^{m+1}\, f(h) + f(0)\right] - \frac{m^2\,\pi^2}{h^2}\, f_s(m) \tag{A.3}$$

Following generalized Hankel transform has been defined by Marchi and Zgrablich [5]

$$\bar{f}(n) = \int_a^b x\, f(x)\; s_p(k_1,k_2,\mu_n x)\, dx \qquad \text{(A.4)}$$

where $$s_p(k_1,k_2,\mu_n x) = J_p(\mu_n x) \left[G_p(k_1,\mu_n a) + G_p(k_2,\mu_n b)\right]$$

$$- G_p(\mu_n x) \left[J_p(k_1,\mu_n a) + J_p(k_2,\mu_n b)\right] \qquad \text{(A.5)}$$

$$J_p(k_i,\mu_n x) = J_p(\mu_n x) + k_i\, \mu_n\, J'_p(\mu_n x) \qquad \text{(A.6)}$$

$$G_p(k_i,\mu_n x) = G_p(\mu_n x) + k_i\, \mu_n\, G'_p(\mu_n x) \qquad \text{(A.7)}$$

$J_p(\mu_n x)$ and $G_p(\mu_n x) = \frac{1}{2}\, cosec\, (P\pi) \left[J_{-p}(\mu_n x) - e^{-ip\pi}\, J_p(\mu_n x)\right]$ are Bessel functions of first and second kind [3] respectively of order p :

μ_n are the positive roots of the equation

$$J_p(k_1,\mu_n a)\; G_p(k_2,\mu_n b) \quad J_p(k_2,\mu_n b)\; G_p(k_1,\mu_n a) = 0 \qquad \text{(A.8)}$$

The inverse transform is given by

$$f(x) = \sum_{n}^{\infty} \frac{1}{c_n} \bar{f}(n)\, s_p(k_1,k_2,\mu_n x) \tag{A.9}$$

where

$$c_n = \frac{1}{2} b^2 \left[s_p^2(k_1,k_2,\mu_n b) - s_{p-1}(k_1,k_2,\mu_n b)\, s_{p+1}(k_1,k_2,\mu_n b) \right]$$

$$- \frac{1}{2} a^2 \left[s_p^2(k_1,k_2,\mu_n a) - s_{p-1}(k_1,k_2,\mu_n a)\, s_{p+1}(k_1,k_2,\mu_n a) \right] \tag{A.10}$$

One of the basic properties of the above transform is

$$\int_a^b x \left(\frac{\partial^2 f}{\partial x^2} + \frac{1}{x} \frac{f}{x} - \frac{p^2}{x^2} f \right) s_p(k_1,k_2,\mu_n x)\, dx$$

$$= \frac{b}{k_2} s_p(k_1,k_2,\mu_n b) \left[f + k_2 \frac{\partial f}{\partial x} \right]_{x=b}$$

$$- \frac{a}{k_1} s_p(k_1,k_2,\mu_n a) \left[f + k_1 \frac{\partial f}{\partial x} \right]_{x=a} - \mu_n^2 \bar{f}(n) \tag{A.11}$$

The sine transform [8,9] of a function $f(x)$ is defined as

$$f_{s_1}(m) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin mx\, dx \tag{A.12}$$

where $$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_{s_1}(m) \sin mx \, dm \tag{A.13}$$

It has a following property

$$\sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\partial^2 f}{\partial x^2} \sin mx \, dx = -m^2 f_{s_1}(m) + m f(0) \tag{A.14}$$

REFERENCES

[1] ARPACI, V.S.: *Conduction Heat Transfer*, Addison-Wesley, Reading, Mass., 1966.

[2] CARSLAW, H.S. and JAEGER, J.C.: *Conduction of Heat in Solids*, Oxford University Press, London, 1959.

[3] ERDELYI, A. et al.: *Higher Transcendental Functions*, Vol.I - II, Mc Graw-Hill, New York, 1953.

[4] ERDELYI, A. et al. : *Tables of Integral Transforms*, Vol.I, Mc Graw-Hill, New York, 1954.

[5] MARCHI, E. and ZGRABLICH, G.: "Heat Conduction in a Hollow Cylinder with Radiation", *Proc. Edinburgh Math. Soc.*, Vol.14, N°2, 1964, pp.159-64.

[6] MATHUR, S.L. : "On Heat Conduction I - II", *Indian Journal of Physics*, Vol.45, 1971, pp.18 - 27.

[7] MEHTA, D.K.: "Some Time Reversal Problems of Heat Conduction", *Proc. Natn. Acad. of Sci.*, India, Vol.39, 1969, pp.397-404 (A).

[8] SNEDDON, I.N.: *Fourier Transforms*, Mc Graw-Hill, New York, 1951.

[9] SNEDDON, I.N. : *The Use of Integral Transforms* , Tata - Mc Graw Hill, India, 1974.