AN EXPANSION FORMULA FOR THE H-FUNCTION OF TWO VARIABLES

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ABSTRACT

The object of this note is to evaluate an integral associated with Bessel polynomials and the H-function of two variables and to apply it in proving an expansion formula for the H-function of two variables in series of product of the Bessel polynomials and a related H-function of two variables. The results obtained are of general character and the integrals & series expansions associated with the special functions of Mathematical Physics & Chemistry can be derived as special cases.

RESUMEN

El objeto de este trabajo es evaluar una integral asociada con polinomios de Bessel y la función H- de dos variables, y aplicarla para establecer una fórmula de expansión para la función H- de dos variables en serie de productos de polinomios de Bessel y la función H- relacionada. Los resultados obtenidos son de carácter general y algunas integrales y series asociadas con funciones especiales de Matemáticas, Física y Química, pueden ser derivadas como casos especiales.

1. INTRODUCTION

The Bessel Polynomials arise from the solution of the classical wave equation in spherical coordinates, namely

$$x^2 \frac{d^2y}{dx^2} + (a\,x + b) \frac{dy}{dx} - k\,(k + a - 1)\,y = 0$$

Krall and Frink [6] has defined these polynomials in terms of the hypergeometric series in the form

$$y_k(x\,,\,a\,;\,b) = {}_2F_0\left(-k\,,\,a + k - 1\,;\,-\frac{x}{b}\right) .$$

Certain properties of these polynomials have been studied by Krall and Frink [6] , Agarwal [1] , Al-Salam [2] , Ragab [12], Hamza [4,5] and Saxena and Hamza [15,16].

The H-function of two variables introduced by Munot and Kalla [10], following the notation of Saxena [14] is defined and represented as follows:

$$H^{L,N,N_1,M,M_1}_{E,(A:C),F,(B:D)}\left[\begin{matrix} x \\ y \end{matrix}\middle|\begin{matrix} (e,\theta;\theta') \\ (a,\alpha)\,;\,(c,\gamma) \\ (f,\phi:\phi') \\ (b,\beta)\,;\,(d,\delta) \end{matrix}\right]$$

$$= -\frac{1}{4\pi^2}\int_{L_1}\int_{L_2} X_1(\xi)\,X_2(\eta)\,X_3(\xi,\eta)\,x^{\xi}\,y^{\eta}\,d\xi\,d\eta \tag{1.1}$$

where

$$X_1(\xi) = \frac{\prod_{1}^{M} \Gamma(b_j - \beta_j \xi) \prod_{1}^{N} \Gamma(1 - a_j + \alpha_j \xi)}{\prod_{M+1}^{B} \Gamma(1 - b_j + \beta_j \xi) \prod_{N+1}^{A} \Gamma(a_j - \alpha_j \xi)} \tag{1.2}$$

$$X_2(\eta) = \frac{\prod_{1}^{M_1} \Gamma(d_j - \delta_j \eta) \prod_{1}^{N_1} \Gamma(1 - c_j + \gamma_j \eta)}{\prod_{M_1+1}^{D} \Gamma(1 - d_j + \delta_j \eta) \prod_{N_1+1}^{C} \Gamma(c_j - \gamma_j \eta)} \tag{1.3}$$

$$X_3(\xi,\eta) = \frac{\prod_{1}^{L} \Gamma(1 - e_j + \theta_j \xi + \theta'_j \eta)}{\prod_{L+1}^{E} \Gamma(e_j - \theta_j \xi - \theta'_j \eta) \prod_{1}^{F} \Gamma(1 - f_j + \phi_j \xi + \phi'_j \eta)} \tag{1.4}$$

x, y, are not zero and an empty product is interpreted as unity.

The contours L_1 & L_2 run from $-i\infty$ to $+i\infty$ in ξ and η planes respectively, with identations, if necessary to ensure that poles of

$$\Gamma(b_j - \beta_j \xi) \quad (j = 1,\dots,M),$$

$$\Gamma(d_j - \delta_j \eta) \quad (j = 1,\dots,M_1)$$

are separated from the poles of

$\Gamma(1 - e_j + \theta_j + \theta'_j \eta)$, $(j = 1,\dots,L)$;

$\Gamma(1 - a_j + \alpha_j \xi)$, $(j = 1,\dots,N)$ and $\Gamma(1 - c_j + \gamma_j \eta)$, $(j = 1,\dots,N_1)$.

The integral in (1.1) converges if

$$\psi_1 = \sum_1^L \theta_j - \sum_1^E \theta_j + \sum_1^N \alpha_j - \sum_{N+1}^A \alpha_j + \sum_1^M \beta_j - \sum_{M+1}^B \beta_j - \sum_1^F \phi_j > 0 ,$$

$$\psi_2 = \sum_1^L \theta_j' - \sum_{L+1}^E \theta_j' + \sum_1^{N_1} \gamma_j - \sum_{N_1+1}^C \gamma_j + \sum_1^{M_1} \delta_j - \sum_{M_1+1}^D \delta_j - \sum_1^F \theta_j' > 0$$

with $|arg\ x| < \frac{1}{2}\pi\psi_1$ and $|arg\ y| < \frac{1}{2}\pi\psi_2$.

This function has also been defined and studied independently in a slightly variant forms by Pathak [11] , Verma [17] and Mittal & Gupta [9], but in essence the function remains same.

In the present note we evaluate an integral associated with Bessel polynomials the H-function of two variables & apply it in deriving an expansion formula for H-function of two variables.

2. INTEGRAL

The formula to be proved here is

$$\int_0^\infty t^{r-1} e^{-t} y_k(1,a;t)\, H^{L,N,N_1,M,M_1}_{E,(A:C),F,(B,D)}\left[\begin{matrix} x t^\lambda \\ y t^\mu \end{matrix}\right] dt$$

$$= H^{L+2,N,N_1,M,M_1}_{E+2,(A:C),F+1,(B:D)}\left[\begin{matrix} x \\ y \end{matrix}\ \middle|\ z^* \right] \qquad (2.1)$$

where

$$Z^* = \left| \begin{matrix} (e, \theta : \theta'), \ (1+k-\gamma, \lambda : \mu), \ (2-\gamma-a-k, \lambda : \mu) \\ (a, \alpha), \ (c, \gamma) \\ (f, \phi : \phi'); \ (2-\gamma-a, \lambda : \mu) \\ (b, \beta); \ (d, \delta) \end{matrix} \right|$$

(2.2)

where $R(\gamma - k + \lambda \, min \; b_j/\beta_j + \mu \, min \; d_r/\delta_r) > 0$

$R(\gamma + a + k + \lambda \, min \; b_j/\beta_j + \mu \, min \; d_r/\delta_r) > 1$

$(j = 1, \ldots, M \;;\; r = 1, \ldots, M_1)$;

$|arg\ x| < \frac{1}{2} \pi \psi_1 \;,\; |arg\ y| < \frac{1}{2} \pi \psi_2 \;,\; \psi_1 > 0 \;,\; \psi_2 > 0.$

(2.1) can be established on substituting the value of the H-function of two variables in terms of the contour integral (1.1), interchanging the order of integration, which is obviously justified in view of the conditions stated with the result, evaluating the t-integral from Hamza's formula [4], namely

$$\int_0^\infty t^{\gamma-1} e^{-t} Y_k(1, a; t) \, dt = \Gamma(\gamma - k) \; (\gamma + a - 1)_k \;, \tag{2.3}$$

where

$$R(\gamma - k) > 0 \quad , \quad R(\gamma + a + k) > 1$$

and applying the definition (1.1).

3. AN EXPANSION FORMULA

The following series expansion will be developed here

$$t^{w}\, H^{L,N,N_1,M,M_1}_{E,(A:C),F,(B:D)}\left[\begin{matrix} x\, t^{\lambda} \\ y\, t^{\mu}\end{matrix}\left|\begin{matrix}(e\,,\theta:\theta') \\ (a\,,\alpha)\;;\;(c\,,\gamma) \\ (f\,,\phi:\phi') \\ (b\,,\beta)\;;\;(d\,,\delta)\end{matrix}\right.\right] =$$

$$= \sum_{r=0}^{\infty} \frac{1}{r!\;\Gamma(2-a-r)}\, Y_r(1\,,a\,;t)$$

$$H^{L+2,N,N_1,M,M_1}_{E+2,(A:C),F+1,(B:D)}\left[\begin{matrix} x \\ y\end{matrix}\,\middle|\, Z_r\right] \;, \qquad (3.1)$$

where

$$Z_r = \left|\begin{matrix}(e\,,\theta:\theta')\;,\;(-r+a-w-1\,,\lambda:\mu)\;,\;(-r-w\,,\lambda:\mu) \\ (a\,,\alpha)\;;\;(c\,,\gamma) \\ (f\,,\phi:\phi')\;;\;(-w\,,\lambda:\mu) \\ (b\,,\beta)\;;\;(d\,,\delta)\end{matrix}\right|$$

$R(w + min\; b_j/\beta_j + min\; d_r/\delta_r > 0$ for $j = 1,\ldots,M$ and $r = 1,\ldots,M_1$;

$|arg\; x| < \frac{1}{2}\pi\,\psi_1$ and $|arg\; y| < \frac{1}{2}\pi\,\psi_2$; $\psi_1 > 0$, $\psi_2 > 0$, ψ_1 and ψ_2 are defined in (2.1).

Proof. Let

$$f(t) = t^{w}\, H\left[\begin{matrix} x\, t^{\lambda} \\ y\, t^{\mu}\end{matrix}\right]$$

$$= \sum_{R=0}^{\infty} A_R\, Y_R(1\,,a\,;t)\,. \qquad (3.2)$$

The equation (3.2) is valid , since $f(t)$ is continuous and of bounded variation in the open interval $(0,\infty)$, where $w \geqslant 0$.

Multiplying both sides of (3.2) by $t^{1-a}\, exp(-t)\, y_u(1,a;t)$ and integrating with respect to t from 0 to ∞, we see that

$$\int_0^\infty t^{w+1-a}\, e^{-t}\, y_u(1\,,a\,;t)\; H\begin{bmatrix} x\,t^{\lambda} \\ y\,t^{\mu} \end{bmatrix} dt$$

$$= \int_0^\infty \sum_{R=0}^{\infty} A_R\, exp(-t)\, t^{1-a}\, y_R(1\,,a\,;t)\, y_u(1\,,a\,;t)\, dt.$$

On using (2.1) and the orthogonality property of the Bessel polynomials given by Hamza [5], namely

$$\int_0^\infty x^{1-a}\, e^{-x}\, y_n(1\,,a\,;x)\; y_m(1\,,a\,;x)\; dx$$

$$= \begin{cases} 0 & ,\ \text{if}\ \ m \neq n \\ n!\ \Gamma(2-a-n) & ,\ \text{if}\ \ m = n\,, \end{cases} \tag{3.3}$$

we observe that

$$A_u\ (u)!\ \Gamma(2-a-u)$$
$$= H^{L+2,N,N_1,M,M_1}_{E+2,[A:C],F+1,[B:D]}\begin{bmatrix} x \\ y \end{bmatrix} z_u \Bigg] . \tag{3.4}$$

(3.1) now readily follows from (3.4).

4. SPECIAL CASES

(i) If we set $L = E = F = 0$ and employ the identity [14,p.187], we find that

$$t^{w}\, H^{M,N}_{A,B}\left(x\, t^{\lambda}\,\middle|\, \begin{matrix}(a,\alpha)\\ (b,\beta)\end{matrix}\right) H^{M_1,N_1}_{C,D}\left(y\, t^{\mu}\,\middle|\, \begin{matrix}(c,\gamma)\\ (d,\delta)\end{matrix}\right)$$

$$= \sum_{r=0}^{\infty} \frac{1}{r!\,\Gamma(2-a-r)}\, y_r(1\,,a\,;t)$$

$$H^{2,N,N_1,M,M_1}_{2,(A:C),1,(B:D)}\left[\begin{matrix}x\\ y\end{matrix}\,\middle|\, \begin{matrix}(r+a-w-1\,,\lambda:\mu)\;,\;(-r-w\,,\lambda:\mu)\\ (a\,,\alpha)\;;\;(c\,,\gamma)\\ (-w\,,\lambda:\mu)\\ (b\,,\beta)\;;\;(d\,,\delta)\end{matrix}\right]$$

under the same conditions as given with (3.1).

Since H-function is the most generalized special function, a number of special cases of (3.1) can be easily derived on using the tables of the particular cases of the H-function given in a recent monograph by Mathai and Saxena [8].

(ii) Finally taking $M = B = 1$, $N = A = 0$, $b_1 = 0$, letting $x \to 0$ and $\gamma_i = \delta_j = 1$ for all i,j and using the identity [14,p.187], we obtain a result given recently by Saxena and Hamza [15].

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