Rev. Téc. Ing., Univ. Zulia
Vol. 1, Nº 1, 1978, pp. 63-67

Impreso en Venezuela

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Integral Perturbations in the Theory of Nonlinear Systems of Differential Equations

(Recibido el 13 de Septiembre, 1977)

RESUMEN

El fin de este estudio es investigar algunos problemas acerca de sistemas no-lineales de ecuaciones diferenciales con perturbaciones donde éstas revisten un caracter más general que se consideró previamente.

Las herramientas principales de nuestro análisis son: la fórmula para la variación de constantes desarrollada por Alekseev y una desigualdad integral recientemente establecida por el autor.

SUMMARY

The aim of this paper is to investigate some problems related to perturbed nonlinear systems of differential equations allowing more general perturbations than were previously allowed. The main tools in our analysis are the variation of constants formula developed by Alekseev and the integral inequality recently established by the present author.

1. INTRODUCTION. Many recent papers have dealt with the perturbations of nonlinear systems of differential equations. Brauer [2], Brauer and Strauss [4], Fennell and Proctor [5], Marlin and Struble [6], Pachpatte [7, 8, 9] Strauss [11], Strauss and Yorke [12], and several other authors have considered conditions under which various forms of the stability and asymptotic behavior of a particular solution of the unperturbed system would be preserved under the perturbation. In this paper we wish to study the boundedness, asymptotic behavior, and the rate of growth of the solutions of perturbed nonlinear systems allowing integral perturbations. We are interested in the relations between the solutions of the unperturbed system

$$x' = f(t, x) \tag{1}$$

and the solutions of the perturbed system

$$y' = f(t, y) + g\left(t, y, \int_{t_0}^{t} k(t, s, y)\, ds\right). \tag{2}$$

Here x, y, f, g and k are the elements of R n, an n-dimensional Euclidean space. Let I be the interval $o \leqslant t < \infty$ and Ω be a region in R^n. We shall assume that $f \in C[I \times \Omega, R^n]$ $f_x(t, x) = o$ for $t \geqslant o$ and $f_x(t,x)$ exists and is continuous on I Ω into R^n and that $k \in C[I \times I \times \Omega, R^n]$ and $g \in C[I \times \Omega \times \Omega, R^n]$. Throughout this paper, $x(t) = x(t, t_0, x_0)$ $y(t) = y(t, t_0, y_0)$ will denote unique

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solutions of (1) and (2) respectively, satisfying $x(t_0,t_0,x_0) = x_0$ and $y(t_0,t_0,y_0) = y_0$, and $\Phi(t,t_0,x_0)$ will denote the fundamental matrix solution of the variational equation

$$z' = f_x(t, x(t,t_0,x_0))z \tag{3}$$

such that $\Phi(t_0,t_0,x_0)$ is the identity matrix. We recall that

$$\frac{\partial}{\partial x_0}[x(t,t_0,x_0)] = \Phi(t,t_0,x_0)$$

and that

$$\frac{\partial}{\partial t_0}[x(t,t_0,x_0)] = \Phi(t,t_0,x_0)f(t_0,x_0).$$

We assume that for arbitrary $t_0 \leqslant o$ and $x_0 \in \Omega$ the solution $x(t,t_0,x_0)$ of (1) exists for $t \geqslant o$ and has values in Ω. This, of course, implies that the corrospending matrix $\Phi(t,t_0,x_0)$ exists in the same circumstances. We assume that there exists a subregion Ω_1 of Ω such that for arbitrary $a \in \Omega_1$ and $t_0 \geqslant o$, the solution $x(t,t_0,a)$ of (1) exists on I and has values in Ω. The symbol $|\cdot|$ will denote some convenient norm on R^n as well as a corresponding consistent matrix no.

In most of the earlier works [2], [4], [11], $\Phi(t,t_0,x_0)$ must satisfy a much more stringent hypothesis than is needed here. The most serious of these is $|\Phi(t,t_0,x_0)| \leqslant M$, where $M > o$ and $|x_0|$ sufficiently small. However, in getting the first equation in the proof of Theorem 2 given in [4] (see, also [11]) one must use $|\Phi(t,s,y(s))| \leqslant M$ This implies that it is already known that $|y(s)|$ is uniformly small and thus destroy the ideal nature, if any, of the perturbing terms. The present paper is a step in removing this unpleasant situation and to establish much deeper results under less restrictive conditions.

2. MAIN RESULTS. In this section we state and prove our main results on the behavior of solutions of (2) under suitable assumptions on the perturbation term in (2). To establish our main results in this paper we require the following integral inequality recently established by this author in [10].

LEMMA 1. *Let* u(t), p(t) *and* q(t) *be real-valued nonnegative continuous functions defined on* I, *for which the inequality*

$$u(t) \leqslant u_0 + \int_0^t p(s)u(s)\Big[u(s) + \int_0^s q(\tau)u(\tau)\,d\tau\Big]ds,\ t \in I,$$

holds, where u_0 *is a positive constant. If* $\int_0^t p(\tau)\exp\left(\int_0^\tau q(n)\,dn\right)d\tau < u_0^{-1}$ for all $t \in I$ then

$$u(t) \leqslant u_0 \exp\left(\int_0^t p(s)\left[\frac{u_0\exp\left(\int_0^s q(\tau)\,d\tau\right)}{1 - u_0\int_0^s p(\tau)\exp\left(\int_0^\tau q(n)\,dn\right)d\tau}\right]ds\right),\ t \in I.$$

In our subsequent discussion our interest lies in the following definitions in terms of the behavior of solutions of (1). For similar definitions the reader is referred to [4]

DEFINITION 1. The solution x = o of (1) is said to be globally uniformly stable if there exists a constant M >o such that

$$|x(t,t_0,x_0)| \leqslant M|x_0|,$$

$$t \geqslant t_0 \geqslant o \text{ and } |x_0| < \infty$$

DEFINITION 2. The solution x = o of (1) is said to be *exponentially asymptotically stable* if there exist constants $M > o$, $\alpha > o$ such that

$$|x(t,t_0,x_0)| \leqslant M|x_0|e^{-\alpha(t-t_0)},$$

for all $t \geqslant t_0 \geqslant o$, $|x_0|$ is sufficiently small.

DEFINITION 3. The solution x = o of (1) is said to be *uniformly slowly growing* if, and only if, for every $\alpha > o$ there exists a constant M, possibly depending on α, such that

$$|x(t,t_0,x_0)| \leqslant M|x_0|e^{-\alpha(t-t_0)},$$

for all $t \geqslant t_0 \geqslant o$ and $|x_0| < \infty$

We now state and prove the following theorem on the boundedness of the solutions of (2).

THEOREM 1. *Let the solution* x = o *of* (1) *be globally uniformly stable. Suppose that the fundamental matrix* Φ *of the variational system* (3) *satisfies*

$$|\Phi(t,s,y)| \leqslant N|y|,\ t \geqslant s \geqslant o, \tag{4}$$

where N *is a positive constant,* $y \in \Omega$ *and that the functions* g *and* k *in* (2) *satisfy*

$$|g(t,y,z)| \leqslant p(t)[|y| + |z|],\ t \geqslant o, \tag{5}$$

$$|k(t,s,y)| \leqslant q(s)|y|,\ o \leqslant s \leqslant t\ \infty, \tag{6}$$

where $y, z \in \Omega$; *and* p(t), q(t) *are real-valued non-negative continuous functions defined on* I *such that*

$$\int_{t_0}^{t} N\, p(\tau)\exp\left(\int_{t_0}^{t} q(\eta)\,d\eta\right)d\tau < [M|x_0|]^{-1}$$

for all $t \in I$ *and*

$$\int_{t_0}^{\infty} N\, p(s)\left[\frac{M|x_0|\exp\left(\int_{t_0}^{s} q(\tau)\,d\tau\right)}{1 - M|x|\int_{t_0}^{s} N\, p(\tau)\exp\left(\int_{t_0}^{\tau} q(\eta)\,d\eta\right)d\tau}\right] ds < \infty, \tag{7}$$

where $M > 0$, $x_0 \neq 0$ *are constants and* $x_0 \in \Omega_1$. *Then all solutions of* (2) *are bounded on* I.

Proof. It is known [1, 2] that for x_0 in Ω_1 the solutions of (2) passing through (t_0, x_0) satisfy the integral equation

$$y(t) = x(t) + \int_{t_0}^{t} \Phi(t,s,y(s))\,g\left(s, y(s), \int_{t_0}^{s} k(s,\tau,y(\tau))\,d\tau\right)ds, \tag{8}$$

for all t for which y(t) is in Ω . Using (8), (4), (5) and (6), together with the global uniform stability of the null solution of (1), we obtain

$$|y(t)| \leq M|x| + \int_{t_0}^{t} N\, p(s)|y(s)|\left[|y(s)| + \int_{t_0}^{s} q(\tau)|y(\tau)|\,d\tau\right] ds,$$

This, and the application of Lemma 1 yields

The above estimation in view of the assumption (7) implies the boundedness of y(t) on I, and the theorem is proved.

Our next theorem shows that under some suitable conditions on the perturbation term g and on the function k, all the solutions of (2) approach zero as t

THEOREM 2. *Let the solution* $x \equiv 0$ *of* (1) *be exponentially asymptotically stable. Suppose that the fundamental matrix* Φ *of the variational system* (3) *satisfies*

$$|\Phi(t,s,y)| \leq N|y|e^{-\alpha(t-s)}, \quad t \geq s \geq 0, \tag{9}$$

where N *and* α *are positive constants,* $y \in \Omega$ *and that the functions* g *and* k *in* (2) *satisfy*

$$|g(t,y,z)| \quad p(t)[|y| + |z|], \quad t \quad 0, \tag{10}$$

$$|k(t,s,y)| \leq e^{-\alpha(t-s)}\, q(s)|y|, \quad 0 \leq s \leq t < \infty, \tag{11}$$

where $y, z \in \Omega$; *and* p(t), q(t) *are real -valued non-negative continuous functions defined on* I *such that*

$$\int_{t_0}^{t} N\, p(\tau)e^{-\alpha\tau}\exp\left(\int_{t_0}^{t} q(\eta)\,d\eta\right)d\tau < [M|x_0|e^{-\alpha t_0}]^{-1}$$

for all $t \in I$ *and*

$$\int_{t_0}^{\infty} N\, p(s)e^{-\alpha s}\left[\frac{M|x_0|e^{\alpha t_0}\exp\left(\int_{t_0}^{s} q(\tau)\,d\tau\right)}{1 - M|x|e^{\alpha t_0}\int_{t_0}^{s} N\, p(\tau)e^{-\alpha\tau}\exp\left(\int_{t_0}^{\tau} q(\eta)\,d\eta\right)d\tau}\right] ds < \infty, \tag{12}$$

where $M > 0$, $x_0 \neq 0$ *are constants and* $x_0 \in \Omega_1$, *then all solutions of* (2) *approach zero as* $t \to \infty$.

Proof. It is known, that the solutions of (2) passing through (t_0, x_0), $x_0 \in \Omega_1$ satisfy the integral equation (8) for all t for which y(t) is in Ω . Using (8), (9), (10), (11) together with the exponential asymptotic stability of the null solution of (1), we obtain

$$|y(t)| \leq M|x|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} N|y(s)|e^{-\alpha(t-s)}\, p(s)\left[|y(s)| + \int_{t_0}^{s} e^{-\alpha(s-\tau)} q(\tau)|y(\tau)|\,d\tau\right] ds.$$

The above inequality can be rewritten as

$$|y(t)|e^{-\alpha t} \leq M|x|e^{\alpha t_0} + \int_{t_0}^{t} N\, p(s)e^{-\alpha s}|y(s)|e^{-\alpha s}\left[|y(s)|e^{-\alpha s} + \int_{t_0}^{s} q(\tau)|y(\tau)|e^{\alpha\tau}\,d\tau\right] ds.$$

Now applying Lemma 1 with $u(t) = |y(t)|e^{\alpha t}$, then multiplying by $e^{-\alpha t}$, we obtain

$$|y(t)| \quad M|x_0|\exp\left(\int_{t_0}^{t} N\, p(s)\left[\frac{M|x_0|\exp\left(\int_{t_0}^{s} q(\tau)\,d\tau\right)}{-M|x_0|\int_{t_0}^{s} N\, p(\tau)\exp\left(\int_{t_0}^{\tau} q(\eta)\,d\eta\right)d\tau}\right] ds\right).$$

The above estimation in view of the assumption (12). yields the desired result if we choose M and $|x_0|$ small enough, and the proof of the theorem is complete.

Theorem 3 below demonstrates that the solution of (2) grows more slowly than any positive exponential.

THEOREM 3. *Let the solution* $x \equiv 0$ *of* (1) *be uniformly slowly growing. Suppose the fundamental matrix* Φ *of the variational system* (3) *satisfies*

$$|\Phi(t,s,y)| \leq N|y|e^{\alpha(t-s)}, \quad t \geq s \geq 0, \tag{13}$$

where N *and* α *are positive constants*, $y \in \Omega$ *and that the functions* g *and* k *in* (2) *satisfy*

$$|g(t,y,z)| \leq p(t)[|y| + |z|], \quad t \geq 0, \tag{14}$$

$$|k(t,s,y)| \leq e^{\alpha(t-s)} q(s)|y|, \quad 0 \leq s \leq t \leq \infty, \tag{15}$$

where $y, z \in \Omega$; *and* p(t), q(t) *are real-valued non-negative continuous functions defined on* I *such that*

$$\int_{t_0}^{t} Np(\tau)e^{\alpha\tau}\exp\left(\int_{t_0}^{\tau} q(n)dn\right)d\tau < [M|x_0|e^{-\alpha t_0}]^{-1}$$

for all $t \in I$ *and*

$$\int_{t_0}^{\infty} Np(s)e^{\alpha s}\left[\frac{M|x_0|e^{-\alpha t_0}\exp\left(\int_{t_0}^{s} q(\tau)d\tau\right)}{1 - M|x_0|e^{-\alpha t_0}\int_{t_0}^{s} Np(\tau)e^{-\alpha\tau}\exp\left(\int_{t_0}^{\tau} q(n)dn\right)d\tau}\right] ds < \infty, \tag{16}$$

where $M > 0$, $x_0 \neq 0$ *are constants and* $x_0 \in \Omega_1$, *then all solutions of* (2) *are slowly growing.*

The proof of this theorem follows by the similar argument as in the proof of Theorem 2 with suitable modifications, and hence we omit the details.

As mentioned in the introduction several authors have studied the behavioral relationships between the solutions of (1) and (2) when the integral term in (2) is absent. The type of the hypotheses imposed on the perturbation term in (2) are general enough as compared to those given in [2], [4], [11]. Concerning the fundamental matrix of the variational system (3), the present hypotheses are much less restrictive.

We observe from the proofs of these three theorems that there is no essential difficulty in obtaining analogous results for the perturbed Volterra integrodifferential system of the form

$$y' = f(t,y) + \int_{t_0}^{t} h(t,s,y)ds + g(t,y\int_{t_0}^{t} k(t,s,y)ds),$$

as a perturbation of the nonlinear Volterra integrodifferential system

$$x' = f(t,x) + \int_{t_0}^{t} h(t,s,x)ds,$$

by using the representation formula recently established by Brauer [3] . These theorems will not be given here since there are no new essential ideas to explain.

3. AN EXAMPLE. In this section, we give a simple example to illustrate our Theorem 2. Consider the differential equations

$$x' = -x - e^{-\frac{1}{2}t}x^{-\frac{1}{2}}, \quad t \geq t_0 \geq 0, \quad x(t_0) = x_0 \geq 1, \tag{17}$$

and

$$y' = -y - e^{-\frac{1}{2}t}y^{\frac{1}{2}} + g(t,y, \int_{t_0}^{t} k(t,s,y)ds), \quad t \geq t_0 \geq 0, \quad y(t_0) = x_0, \tag{18}$$

Suppose that the functions g and k in (18) satisfy the hypotheses (10), (11) and (12) of Theorem 2 with α 1. The solution $x(t) = x(t, t_0, x_0)$ of (17) is given by

$$x(t)e^{t} = \left[[x\, e^{t_0}]^{\frac{1}{2}} - \frac{1}{2}(t - t_0)\right]^{2}, \quad t \geq t \geq 0, \tag{19}$$

We observe from (19) that

$$|x(t)| \leq |x|e^{-\alpha(t-t_0)}, \quad t \geq t_0 \geq 0. \tag{20}$$

From (20) it is clear that the null solution of (17) is exponentially asymptotically stable.

Here

$$\Phi(t,t_0\, x_0) = \left[x_0^{\frac{1}{2}}e^{(t-t_0)} - \frac{1}{2}(t-t_0)e^{(t-\frac{1}{2}t_0)}\right]x_0^{-\frac{1}{2}} \tag{21}$$

We observe from (21) that

$$|\Phi(t,t_0,x_0)| \leq |x_0|e^{-t-t_0)}, \quad t \geq t_0 \geq 0,$$

i.e.

$$|\Phi(t,s,y)| \leq |y|e^{(t-s)}, \quad t \geq s \geq 0$$

Clearly the fundamental matrix Φ corresponding the solution $x(t, t_0, x_0)$ of (17) satisfies the hypothesis (8) of Theorem 2. Thus all the hypotheses of Theorem 2 are satisfied and therefore the conclusion of the theorem is true.

REFERENCES

Alekseev, V.M., An estimate for the perturbations of the solutions of ordinary differential equations, *Vestn. Mosk. Ser. I. Math. Meh.*, Vol. 2, 1961, pp. 28 - 36.

Brauer, F., Perturbations of nonlinear systems of differential equations II, *J. Math. Anal. Appl.* Vol., 17, 1967, pp. 418 - 434.

Brauer, F., A nonlinear variation of constants formula for Volterra equations, Math. *Systems Theory*, Vol. 6, 1972, pp. 226 - 234.

Brauer, F. and Strauss, A., Perturbations of nonlinear systems of differential equations III, *J. Math. Anal. Appl.*, Vol. 31, 1970, pp. 37 - 48.

Fennell, R.E. and Proctor, G., On asymptotic behavior on perturbed nonlinear systems, *Proc. Amer. Math. Soc.*, Vol. 31, 1972, pp. 495 - 504.

Marlin, J.A. and Struble, R.A., Asymptotic equivalence of nonlinear systems, *J. Differential Equations*, Vol. 6, 1969, pp. 578 - 596.

Pachpatte, B. G., Stability and asymptotic behavior of perturbed nonlinear systems, *J. Differential Equations*, Vol. 16, 1974, pp. 14 - 25.

Pachpatte, B. G., Integral perturbations of nonlinear systems of differential equations, *Bull. Soc. Math. Grece*, Vol. 14, 1973, pp. 92 - 97.

Pachpatte, B. G., Perturbations of nonlinear systems of differential equations, *J. Math. Anal. Appl.*, Vol. 51, 1975, pp. 550 - 556.

Pachpatte, B. G., On some new integral inequalities for differential and integral equations, *J. Math. Physical. Sci.*, Vol. 10, 1976, pp. 101 - 116.

Strauss, A., On the stability of a perturbed nonlinear system, *Proc. Amer. Math. Soc.*, Vol. 17, 1966, pp. 803 - 807.

Strauss, A. and Yorke, J. A., Perturbation theorems for ordinary differential equations, *J. Differential Equations*, Vol. 3, 1967, pp. 15 - 30.