Divulgaciones Matemáticas Vol. 23-24, No. 1-2 (2022-2023), pp. 82–106
https://produccioncientificaluz.org/index.php/divulgaciones/
DOI: https://doi.org/10.5281/zenodo.11540455

e-ISSN 2731-2437
p-ISSN 1315-2068

Boundary Estimation with the Fuzzy Set Regression Estimator

Estimación Frontera con el Estimador de Regresión con Conjunto Difuso

Jesús A. Fajardo
(jfajardogonzalez@gmail.com;jfajardo@udo.edu.ve)
Departamento de Matemática del Núcleo de Sucre
Universidad de Oriente
Cumaná 6101, República Bolivariana de Venezuela

Abstract

In order to extend the properties of the fuzzy set regression estimation method and provide new results related to the nonparametric regression estimation problems not based on kernels, this paper analyzes the possible boundary effects, if any, of the fuzzy set regression estimator and presents a criterion to remove it. Moreover, a boundary fuzzy set estimator is proposed which is defined as a particular class of fuzzy set regression estimators, where the bias, variance, mean squared error and function that minimizes the mean squared error of the proposed estimator are given. Finally, these theoretical findings are illustrated through some numerical examples, and with one real data example. Simulations show that the proposed estimator has better performance at points near zero in a spread neighborhood of the smoothing parameter, when it is compared with a general boundary kernel regression estimator for the two regression models and two density functions considered. The previously exposed represents the natural extension of the recent results to the boundary fuzzy set density estimator case.

Palabras y frases clave: Fuzzy set regression estimator, boundary estimation.

Resumen

Con el fin de ampliar las propiedades del método de estimación de regresión con conjunto difuso y proporcionar nuevos resultados relacionados con los problemas de estimación no paramétrica de la regresión no basados en núcleos, este artículo analiza los posibles efectos frontera, si los hay, del estimador de regresión con conjunto difuso y presenta un criterio para eliminarlo. Además, se propone un estimador frontera con conjunto difuso el cual se define como una clase particular de estimadores de regresión con conjunto difuso, donde el sesgo, la varianza, el error cuadrático medio y la función que minimiza el error cuadrático medio del estimador propuesto son presentados. Finalmente, estos resultados teóricos se ilustran a través de algunos ejemplos numéricos y con un ejemplo de datos reales. Las simulaciones muestran que el estimador propuesto tiene un mejor desempeño en los puntos cercanos a cero en un entorno disperso del parámetro de suavizado, cuando se compara con un estimador general frontera de la regresión con núcleo para los dos modelos de regresión y las dos funciones de densidad consideradas. Lo expuesto anteriormente representa la extensión

Recibido 04/05/20023. Revisado 14/7/2023. Aceptado 12/12/2023.
MSC (2010): Primary 62G99; Secondary 62G05.
Autor de correspondencia: Jesús Fajardo.

natural de los resultados recientes al caso del estimador frontera de la densidad con conjunto difuso.

Key words and phrases: Estimador de regresión con conjunto difuso, estimación frontera.

1 Introducción

Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be n independent copies of a random vector (X, Y). The regression model is given by

$$Y_i = r(X_i) + \varepsilon_i, \ i = 1, \ldots, n,$$

where the observation errors $\varepsilon_1, \ldots, \varepsilon_n$ are random variables such that

$$\mathbb{E}[\varepsilon_i|X] = 0, \ Var[\varepsilon_i|X] = \sigma^2 < \infty, \ i = 1, \ldots, n,$$

and unknown regression function $r(t) = \mathbb{E}[Y|X = t]$, for $t \in \mathbb{R}$. The main goal of this paper is to introduce a new method to estimate the regression function r at points near zero in a spread neighborhood of the smoothing parameter.

Numerous nonparametric methods have been developed in the literature to estimate r, with independent pairs of data. Nevertheless, most of those estimation methods are not consistent when the support of r has finite endpoints, seriously affecting the overall performance of the implemented estimators. The lack of consistency is a consequence of the so called "boundary effects," where the connection between the estimation methods and boundary effects is reflected in the performances of the proposed estimators for each method. This makes their performances at boundary points usually to differ from the performances at interior points. Theoretically, the convergence rates of the estimators at boundary points are slower than the convergence rates at interior points of the support of r. Strictly speaking, the bias of the estimators is of order $O(b_n)$ instead of $O(b_n^2)$ at boundary points, where b_n is the smoothing parameter or bandwidth ($b_n \to 0$ as $n \to \infty$). This imposes the need to carry out a study to detect whether the boundary effects are present or not in the estimator of any function, since it is not obvious that the behavior of the estimator is the same at both the boundary and the interior points. To remove those boundary effects in kernel regression estimation, a variety of methods have been developed in the literature. An excellent summary of some well-known methods is given in [24]. Finally, it is important to remark here that the above phenomenon, called "boundary effects" in the estimation theory, also affects the fuzzy set regression estimator introduced in [9].

In this paper, a criterion to remove the boundary effects, without boundary modifications, in the nonparametric fuzzy set regression estimator is proposed, obtaining a natural extension of the approach introduced in [8] to the regression case. For this, at each point near 0 in a b_n spread neighborhood, the proposed boundary estimator is defined as a particular class of fuzzy set regression estimators, where the bias, variance and optimal mean squared error (MSE) are given. Moreover, the function that minimizes the MSE is obtained. Simultaneously, extensive simulations are carried out to compare the local MSE of the proposed boundary estimator with the local MSE of the general boundary estimator given in [24] at points near 0 in a b_n spread neighborhood, observing that the local MSE of the proposed boundary estimator is the smallest for the two regression models and the two density functions considered. This reduction shows that the boundary fuzzy set regression estimator has better performance than the estimator studied in [24]. Besides, it is appropriate to note that the above results extend the properties of the fuzzy set regression estimation method, providing new properties related to the nonparametric regression estimation problems not based on kernels.

The particular choice above was based mainly on the results of the simulations obtained in [24], for the two regression models and the two density functions considered in this work, which showed that the general boundary kernel regression estimator defined in the above paper performed quite well when it was compared with both local linear and classical kernel regression estimators. Among other reasons that supported the above particular choice, the theoretical properties that are shared by the boundary estimator defined in [24] and the proposed boundary estimator are highlighted: non-negativity, “natural boundary continuation” and they improve the bias but holding on to the low variances. Moreover, it is worth pointing out that the paper [24] extends the approach introduced in [17] to the regression case, by defining the popular Nadaraya-Watson estimator, [20, 27], in terms of the boundary kernel density estimator given in [17]. It is worth noting that the results of the simulations presented in [17] for the four shapes of densities considered showed that the boundary kernel density estimator introduced in the above work performed quite well when it was compared with the estimators boarded in [16, 29] and its simple modification which allows obtaining the local linear fitting estimator [13,30]. Nonetheless, the results of the simulations obtained in [8], for the four shapes of densities considered in [17], showed that the boundary fuzzy set density estimator performed quite well when it was compared with the boundary kernel estimator defined in [17]. Now, combining this last result with the idea established in [24], it is reasonable to define an estimator of the Nadaraya-Watson type regression function in terms of the boundary fuzzy set density estimator, in order to achieve the objectives emphasized in this paper and to solve the problem proposed in [8]. On the other hand, a literature review on the proposed topic revealed that there is not evidence of publications with respect to the comparison of the performance between other methods and the method introduced in [24]. Besides, the author guarantees a conclusion analogous to the previous one for the fuzzy set regression estimator case. Finally, it is necessary to point out that in the recent works [1–4, 15, 18, 19, 25] the problems of nonparametric regression estimation are studied under specific conditions and new regression estimators are introduced through the approach of each previous work. It should be noted that the method introduced in [15] combines the smoothing spline and kernel functions. Nonetheless, in the papers [1, 3, 4, 18] and [2, 19, 25] both Nadaraya-Watson and local linear estimators are the main actors, respectively. This last point suggests the combination of the approaches in the works [2, 19, 25] and [7] to future research, since in [7] was shown that the fuzzy set regression estimator has better performance than the local linear regression smoothers.

This paper is organized as follows. In Section 2, the boundary effects in the fuzzy set regression estimator are studied and the criterion to remove such effects is presented. Moreover, the boundary fuzzy set regression estimator is defined and its asymptotic properties are introduced. Besides, the function that minimizes the MSE of the proposed boundary estimator is calculated. The simulation studies and data analysis are introducen in Sections 3 and 4, respectively. Final comments are given in Section 5.

2 Fuzzy set regression estimator and boundary effects

A study to detect the presence or not of the boundary effects in the estimator of any function is necessary since it is not obvious that the behavior of the estimator can be the same at the boundary points as in the interior points. Consequently, this section analyzes the possible boundary effects, if any, of the fuzzy set regression estimator given in [9], and introduces a criterion to remove it, without boundary modifications. Moreover, the definition of the boundary fuzzy set regression estimator and its asymptotic properties are given. Also, the function that minimizes the MSE of the proposed boundary estimator is obtained.

2.1 Fuzzy set estimator of the regression function

It is important to emphasize that the fuzzy set regression estimation method introduced in [9] is based on defining an estimator of the Nadaraya-Watson type for independent pairs of data in terms of the fuzzy set density estimator given in [10]. In order to familiarize the reader with the above method, a general summary of the details that allowed to define the estimator introduced in [10] will be presented.

For independent copies $X_1, \ldots, X_n$ of a random variable X in $\mathbb{R}$, whose distribution $\mathcal{L}(X)$ has a Lebesgue density f_X near some fixed point $x_0 \in \mathbb{R}$, a fuzzy set estimator of f_X, at the point $x_0 \in \mathbb{R}$, is defined in [11], by means of thinned point processes $N_n^{\varphi_n}$, a process framed inside the theory of the point processes, as follows

$$\hat{\theta}_n(x_0) = \frac{N_n^{\varphi_n(t)}(\mathbb{R})}{n\, a_n} = \frac{1}{n\, a_n} \sum_{i=1}^{n} U_i, \quad t \in \mathbb{R},$$

where

$$N_n^{\varphi_n} = \frac{1}{n\, a_n} \sum_{i=1}^{n} U_i\, \varepsilon_{X_i},$$

$\varphi_n(t) = \mathbb{P}(U_i = 1 | X_i = t)$, ε_X is the random Dirac measure, $a_n > 0$ is a smoothing parameter or bandwidth such that $a_n \to 0$ as $n \to \infty$, and the random variables U_i, $1 \leq i \leq n$, are independent with values in $\{0, 1\}$, which determines whether X_i belongs to the neighborhood of x_0 or not. See e.g. [21], for more details on the theory of the point processes and thinned point processes. In [11], only the asymptotic efficiency within the class of all estimators that are based on randomly selected points from the sample $X_1, \ldots, X_n$ was proved. Efficiency was established using LeCam's LAN theory. Although the almost sure and uniform convergence properties over a compact subset on $\mathbb{R}$ are not studied, the pointwise convergence in law whose distribution limit has an asymptotic variance that depends only of $f_X(x_0)$ is proposed. On the other hand, it is opportune to point out that the random variables that define the estimator $\hat{\theta}_n$ do not possess, for example, precise functional characteristics in regards to the point of estimation. This absence of functional characteristics complicates the evaluation of the estimator using a sample. Thus, the simulations to estimate the density function will be more complicated. However, to overcome the difficulties presented by the estimator introduced in [11], a particular case of the above estimator was defined in [10].

Let X be a real random variable whose distribution $\mathcal{L}(X)$ has a Lebesgue density f_X near some fixed point $x_0 \in \mathbb{R}$. For each measurable Borel function $\varphi : \mathbb{R} \to [0, 1]$ and each random variable V, uniformly distributed on $[0, 1]$ and independent of X, the random variable $\mathbb{1}_{[0,\varphi(X)]}(V)$ satisfies $\varphi(t) = \mathbb{P}(\mathbb{1}_{[0,\varphi(X)]}(V) = 1 | X = t)$. This simple observation allows us to construct a fuzzy set estimator of f_X to estimate $f_X(x_0)$, which satisfies the conditions required in [11]. Moreover, as the local behavior of the distribution of X will be evaluated, it is obvious that only observations X_i in a neighborhood of x_0 can reasonably contribute to the estimation of $f_X(x_0)$.

Let $X_1, \ldots, X_n$ be an independent random sample of X. Let $V_1, \ldots, V_n$ be independent random variables uniformly distributed on $[0,1]$ and independent of $X_1, \ldots, X_n$. Let $\mathbb{I}_{\left[0,\varphi\left(\frac{X_i - x_0}{b_n}\right)\right]}(V_i)$ be random variables where $b_n > 0$ is a smoothing parameter or bandwidth such that $b_n \to 0$ as $n \to \infty$. For each $t \in \mathbb{R}$, one obtains that

$$\varphi\left(\frac{t - x_0}{b_n}\right) = \mathbb{P}\left(\mathbb{I}_{\left[0,\varphi\left(\frac{X_i - x_0}{b_n}\right)\right]}(V_i) = 1 \middle| X_i = t\right),$$

where $\varphi_n(t) = \varphi\left(\frac{t-x_0}{b_n}\right)$ is a Markov kernel (see [21], Section 1.4). Thus, for independent copies (X_i, V_i), $1 \leq i \leq n$, of (X, V), the thinned point process is defined as follows

$$N_n^{\varphi_n}(\cdot) = \sum_{i=1}^{n} \mathbb{I}_{\left[0,\varphi\left(\frac{X_i - x_0}{b_n}\right)\right]}(V_i)\, \varepsilon_{X_i}(\cdot),$$

with underlying point process $N_n(\cdot) = \sum_{i=1}^{n} \varepsilon_{X_i}(\cdot)$ and a thinning function φ_n (see [21], Section 2.4), where ε_X is the random Dirac measure.

On the other hand, observe that the set of observations or the events $\{X_i = t\}$, $t \in \mathbb{R}$, in a neighborhood of x_0 can now be described by the thinned point process $N_n^{\varphi_n}$, where $\mathbb{I}_{\left[0,\varphi\left(\frac{X_i - x_0}{b_n}\right)\right]}(V_i)$ decides, whether X_i belongs to the neighborhood of x_0 or not. Precisely, $\varphi_n(t)$ is the probability that the observation $X_i = t$ belongs to the neighborhood of x_0. Note that this neighborhood is not explicitly defined, but it is actually a fuzzy set in the sense of the paper [28], given its membership function φ_n. The thinned process $N_n^{\varphi_n}$ is therefore a fuzzy set representation of the data (see [11], Section 2).

Next, the fuzzy set density estimator introduced in [10] is presented, which is a particular case of the estimator proposed in [11].

Definition 1. Let $X_1, \ldots, X_n$ be an independent random sample of X. Let $V_1, \ldots, V_n$ be independent random variables uniformly distributed on $[0,1]$ and independent of $X_1, \ldots, X_n$. Let φ be such that $a_n = b_n \int \varphi(u)\, du$ and $0 < \int \varphi(u)\, du < \infty$. Then, the fuzzy set estimator of the density function f_X at the point $x_0 \in \mathbb{R}$ is defined as

$$\hat{\vartheta}_n(x_0) = \frac{1}{na_n} \sum_{i=1}^{n} \mathbb{I}_{\left[0,\varphi\left(\frac{X_i - x_0}{b_n}\right)\right]}(V_i) = \frac{1}{na_n}\, \tau_n(x_0). \tag{1}$$

Observe that estimator (1) can be written in terms of a fuzzy set representation of the data, since $\hat{\vartheta}_n(x_0) = (na_n)^{-1} N_n^{\varphi_n}(\mathbb{R})$. This equality justifies the fuzzy set term of estimator (1), where the thinning function φ_n can be used to select points of the sample with different probabilities, in contrast to the kernel estimator, which assigns equal weight to all points of the sample. Moreover, it is important to highlight that (1) is of easy practical implementation and the random variable $\tau_n(x_0)$ is binomial $\mathcal{B}(n, \alpha_n(x_0))$ distributed with

$$\alpha_n(x_0) = \mathbb{E}\left[\mathbb{I}_{\left[0,\varphi\left(\frac{X - x_0}{b_n}\right)\right]}(V)\right] = \mathbb{P}\left(\mathbb{I}_{\left[0,\varphi\left(\frac{X - x_0}{b_n}\right)\right]}(V) = 1\right) = \mathbb{E}\left[\varphi\left(\frac{X - x_0}{b_n}\right)\right]. \tag{2}$$

In what follows, it will be assumed that $\alpha_n(x_0) \in (0,1)$. Besides, throughout this article φ is the thinning function that characterizes to estimator (1), and the abbreviation "w.t.f." will be used to the phrase "with thinning function." See [6] and [10], for more details on the asymptotic and convergence properties of estimator (1).

Next, the regression estimator of the Nadaraya-Watson type introduced in [9], for independent pairs of data, is presented, which it is defined in terms of estimator (1).

Definition 2. Let $((X_1, Y_1), V_1), \ldots, ((X_n, Y_n), V_n)$ be independent copies of a random vector $((X, Y), V)$, where $V_1, \ldots, V_n$ are independent random variables uniformly distributed on $[0, 1]$, and independent of $(X_1, Y_1), \ldots, (X_n, Y_n)$. Let φ be such that $a_n = b_n \int \varphi(u)\, du$ and $0 < \int \varphi(u)\, du < \infty$. Then, the fuzzy set estimator of the regression function r at the point $x_0 \in \mathbb{R}$ is defined as

$$\hat{r}_n(x_0) = \begin{cases} \dfrac{\sum_{i=1}^{n} Y_i \, \mathbb{1}_{\left[0, \varphi\left(\frac{X_i - x_0}{b_n}\right)\right]}(V_i)}{\tau_n(x_0)} & \text{if} \quad \tau_n(x_0) \neq 0 \\ \\ 0 & \text{if} \quad \tau_n(x_0) = 0, \end{cases} \tag{3}$$

where τ_n is given in (1).

As in the case of estimator (1), the function φ characterizes estimator (3). See [5] and [9], for more details on the asymptotic and convergence properties of estimator (3).

2.2 The boundary problem of the fuzzy set regression estimator and its boundary estimator

In order to provide new results related to the nonparametric regression estimation problems not based on kernels, and to ensure the natural extension of the approach to the boundary fuzzy set density estimation case, this section studies the behavior of estimator (3), at the particular points 0 and $x \in (0, b_n]$, under the following conditions:

C1 Functions f_X and r are at least twice continuously differentiable in a neighborhood of $z \in [0, \infty)$.

C2 $f_X(z) > 0$.

C3 Sequence b_n satisfies: $b_n \to 0,\ nb_n \to \infty,$ as $n \to \infty$.

C4 Function φ is symmetrical regarding zero, has compact support on $[-B, B]$, $B > 0$, and it is continuous at 0 with $\varphi(0) > 0$.

C5 There exists $M > 0$ such that $|Y| < M$ $a.s.$

C6 Function $\phi(z) = \mathbb{E}[Y^2 | X = z]$ is at least twice continuously differentiable in a neighborhood of $z \in [0, \infty)$.

Next, the results associated with the behavior of the estimator

$$\hat{g}_n(z) = \frac{1}{na_n} \sum_{i=1}^{n} Y_i \, \mathbb{1}_{\left[0, \varphi\left(\frac{X_i - z}{b_n}\right)\right]}(V_i), \ \ z \in [0, \infty), \tag{4}$$

at the particular points 0 and $x \in (0, b_n]$, are presented. Here, $g(z) = r(z) f_X(z)$, $a_n = b_n \int \varphi(u)\, du$ and $0 < \int \varphi(u)\, du < \infty$. Moreover, the function ψ will be defined as $\psi(u) = \frac{\varphi(u)}{\int \varphi(u)\, du} \mathbb{I}_{[-B,\, B]}(u)$, and as each $x \in (0, b_n]$ has the form $x = c\, b_n$, $0 < c \leq 1$, in that which follows, to simplify the notation, x will be written instead of $c\, b_n$.

Theorem 1. *Under the conditions* $(C1) - (C4)$*, we have*

$$\mathbb{E}\left[\hat{g}_n(0)\right] = g(0) + \frac{b_n^2}{2} g''(0) \int u^2 \psi(u)\, du + o\left(b_n^2\right).$$

Proof. Replacing z with 0 into (4), given that V and (X, Y) are independent, and taking the conditional expectation with respect to $X = 0$ and $V = v$, we obtain

$$\mathbb{E}\left[\hat{g}_n(0)\right] = a_n^{-1} \int_{-\infty}^{\infty} \varphi\left(\frac{u}{b_n}\right) g(u) du.$$

Combining $(C1)$ and $(C4)$, we can write the above equality as

$$\mathbb{E}\left[\hat{g}_n(0)\right] = \int_{-\frac{B}{b_n}}^{B} \psi(u)\, g\left(u b_n\right) du.$$

As $(C1)$ allows us to make a the Taylor expansion of function g in the neighborhood of 0, we can rewrite the above equality as

$$\mathbb{E}\left[\hat{g}_n(0)\right] = \int_{-\frac{B}{b_n}}^{B} \left[g(0) + b_n\, g'(0) u + \frac{b_n^2}{2} u^2 g''(\lambda u b_n)\right] \psi(u)\, du. \tag{5}$$

Moreover, $(C3)$ allows us to suppose, without loss of generality, that $b_n < 1$. Now, we can guarantee that for $B > 0$

$$\frac{B}{b_n} > B. \tag{6}$$

Combining $(C3)$, (5) and (6), we have

$$\mathbb{E}\left[\hat{g}_n(0)\right] = g(0) + \frac{b_n^2}{2} \int u^2 \left[g''(0) + \left(g''(\lambda u b_n) - g''(0)\right)\right] \psi(u)\, du. \tag{7}$$

Now combining $(C1)$, $(C3)$ and $(C4)$, we obtain

$$\int u^2 \left[g''(\lambda u b_n) - g''(0)\right] \psi(u)\, du = o(1). \tag{8}$$

The result now follows by combining (7) and (8). □

Corollary 1. *Under the conditions* $(C1) - (C4)$*, we have*

$$\mathbb{E}\left[\hat{\vartheta}_n(0)\right] = f_X(0) + \frac{b_n^2}{2} f_X''(0) \int u^2 \psi(u)\, du + o\left(b_n^2\right).$$

Proof. Setting $Y = 1$, we have $g(0) = f_X(0)$ and $\mathbb{E}[\hat{g}_n(0)] = \mathbb{E}[\hat{\vartheta}_n(0)]$. From Theorem 1, we obtain the result. □

Theorem 2. *Under the conditions* $(C1) - (C4)$*, we have*

$$\mathbb{E}\left[\hat{g}_n(x)\right] = \int_{-c}^{B} \left[g(0) + b_n\, g'(0)u + \frac{b_n^2}{2}\, g''(0)u^2\right] \psi(u)\, du + o\left(b_n^2\right).$$

Proof. Replacing z with x into (4), using that V and (X, Y) are independent, and taking the conditional expectation with respect to $X = x$ and $V = v$, we obtain

$$\mathbb{E}\left[\hat{g}_n(x)\right] = a_n^{-1} \int_{-\infty}^{\infty} \varphi\left(\frac{u - x}{b_n}\right) g(u)\, du.$$

Combining $(C1)$ and $(C4)$, we can write the above equality as

$$\mathbb{E}\left[\hat{g}_n(x)\right] = \int_{-\frac{x}{b_n}}^{B} \psi(u)\, g\left(x + u b_n\right)\, du.$$

As $(C1)$ allows us to make a Taylor expansion of function g in the neighborhood of x, we can rewrite the above equality as

$$\mathbb{E}\left[\hat{g}_n(x)\right] = \int_{-c}^{B} \left[g(x) + b_n g'(x)u + \frac{b_n^2}{2} u^2 g''(x + \lambda u b_n)\right] \psi(u)\, du. \tag{9}$$

Now combining $(C1)$ with $(C3)$, and $(C3)$ with $(C4)$, we obtain

$$h(x) = h(0) + o\left(1\right), \text{ for } h = g,\, g', \tag{10}$$

and

$$\int_{-c}^{B} u^2 \left[g''(x + \lambda u b_n) - g''(0)\right] \psi(u) du = o\left(1\right). \tag{11}$$

The result now follows by combining (9), (10) and (11). □

Corollary 2. *Under the conditions* $(C1) - (C4)$*, we have*

$$\mathbb{E}\left[\hat{\vartheta}_n(x)\right] = \int_{-c}^{B} \left[f_X(0) + b_n\, f_X'(0)u + \frac{b_n^2}{2}\, f_X''(0)u^2\right] \psi(u)\, du + o\left(b_n^2\right).$$

Proof. Setting $Y = 1$, we have $g(x) = f_X(x)$ and $\mathbb{E}[\hat{g}_n(x)] = \mathbb{E}[\hat{\vartheta}_n(x)]$. From Theorem 2, we obtain the result. □

The above results allow us to guarantee that both (1) and (4) estimators do not present the boundary effects at 0, and they will present the boundary effects at x when $h'(0) \int_{-c}^{B} u\, \psi(u)\, du \neq 0$, for $h = f_X, g$ and for each $c \in (0, 1]$. Moreover, it is important to remark here that Corollaries 1 and 2 are Theorems 1 and 2 in [8].

On the other hand, it is probable enough that estimator (3) presents the above behavior at the particular points 0 and $x \in (0, b_n]$: estimator (3) does not present the boundary effects at 0 and it will present the boundary effects at x when $r'(0) \int_{-c}^{B} u\, \psi(u)\, du \neq 0$, for each $c \in (0, 1]$. The following two theorems allow to confirm the above suspicion.

Theorem 3. *Under the conditions $(C1)-(C5)$, we have*

$$\mathbb{E}\left[\hat{r}_n(0)\right] = r(0) + \frac{b_n^2}{2}\left[r''(0) + 2\frac{r'(0)f_X'(0)}{f_X(0)}\right]\int u^2\psi(u)\,du + o\left(b_n^2\right).$$

Proof. Let us consider the following decomposition (see, e.g., [12])

$$\hat{r}_n(t) = \frac{\hat{g}_n(t)}{\mathbb{E}\left[\hat{\vartheta}_n(t)\right]}\left(1 - \frac{\hat{\vartheta}_n(t) - \mathbb{E}\left[\hat{\vartheta}_n(t)\right]}{\mathbb{E}\left[\hat{\vartheta}_n(t)\right]}\right) + \frac{\left[\hat{\vartheta}_n(t) - \mathbb{E}\left[\hat{\vartheta}_n(t)\right]\right]^2}{\left[\mathbb{E}\left[\hat{\vartheta}_n(t)\right]\right]^2\left[\hat{r}_n(t)\right]^{-1}}. \tag{12}$$

Replacing z with 0 into (12) and taking the expectation, we obtain

$$\mathbb{E}\left[\hat{r}_n(0)\right] = \frac{\mathbb{E}\left[\hat{g}_n(0)\right]}{\mathbb{E}\left[\hat{\vartheta}_n(0)\right]} - \frac{A_1}{\left[\mathbb{E}\left[\hat{\vartheta}_n(0)\right]\right]^2} + \frac{A_2}{\left[\mathbb{E}\left[\hat{\vartheta}_n(0)\right]\right]^2},$$

where

$$A_1 = \mathbb{E}\left[\left(\hat{\vartheta}_n(0) - \mathbb{E}\left[\hat{\vartheta}_n(0)\right]\right)\hat{g}_n(0)\right] \quad \text{and} \quad A_2 = \mathbb{E}\left[\left(\hat{\vartheta}_n(0) - \mathbb{E}\left[\hat{\vartheta}_n(0)\right]\right)^2\hat{r}_n(0)\right].$$

Combining $(C3)$, (7) and (8), we have

$$\mathbb{E}\left[\hat{g}_n(0)\right] = g(0) + o(1). \tag{13}$$

Now setting $Y = 1$, we can write (4) as

$$\mathbb{E}\left[\hat{\vartheta}_n(0)\right] = f_X(0) + o(1). \tag{14}$$

As the random vectors $((X_i, Y_i), V_i)$, $1 \le i \le n$, are identically distributed, we have

$$\begin{aligned} A_1 &= Cov\left[\hat{g}_n(0), \hat{\vartheta}_n(0)\right] = \frac{1}{(na_n)^2}\sum_{i=1}^n Cov\left[Y_iU_i, U_i\right] = \frac{1}{na_n}\mathbb{E}\left[\frac{YU}{a_n}\right] - \frac{1}{n}\mathbb{E}\left[\frac{YU}{a_n}\right]\mathbb{E}\left[\frac{U}{a_n}\right] \\ &= \frac{1}{na_n}\mathbb{E}\big[\hat{g}_n(0)\big] - \frac{1}{n}\mathbb{E}\big[\hat{g}_n(0)\big]\mathbb{E}\big[\hat{\vartheta}_n(0)\big], \end{aligned}$$

where $U = \mathbb{1}_{[0,\varphi(\frac{X}{b_n})]}(V) = U^2$. Combining $(C3)$, (13) and (14), we obtain

$$A_1 = \frac{1}{na_n}\left[g(0) + o(1)\right] - \frac{1}{n}\left[g(0) + o(1)\right]\left[f_X(0) + o(1)\right] = \frac{1}{na_n}g(0) + o\left(\frac{1}{nb_n}\right).$$

On the other hand, by $(C5)$ there exists $C > 0$ such that $|\hat{r}_n(0)| \le C$. Thus, we can write

$$|A_2| \le C\,\mathbb{E}\left[\left[\hat{\vartheta}_n(0) - \mathbb{E}[\hat{\vartheta}_n(0)]\right]^2\right] = \frac{C}{na_n^2}\left(\mathbb{E}\left[U^2\right] - \left(\mathbb{E}\left[U\right]\right)^2\right) = \frac{C}{na_n}\mathbb{E}\left[\frac{U}{a_n}\right]\left(1 - \mathbb{E}\left[U\right]\right).$$

Now setting $Y = 1$ and combining with (7), we can write

$$\mathbb{E}\left[\frac{U}{a_n}\right] = \mathbb{E}\left[\hat{\vartheta}_n(0)\right] = f_X(0) + \frac{b_n^2}{2}\int u^2\Big[f_X''(0)\left(f_X''(\lambda u b_n) - f_X''(0)\right)\Big]\psi(u)\,du.$$

Moreover, $(C1)$ implies that f_X'' is bounded in the neighborhood of 0, and $(C3)$ allows us to suppose that $b_n < 1$. Then we can bound $\mathbb{E}[\frac{U}{a_n}]$. Besides, by (2) we can bound $(1-\mathbb{E}[U])$. Therefore, $A_2 = O(\frac{1}{nb_n})$. Now, we can write

$$\frac{A_1}{\left(\mathbb{E}\left[\hat{\vartheta}_n(0)\right]\right)^2} = \left(\frac{1}{(f_X(0))^2} + o(1)\right)\left(\frac{1}{nb_n}\right) g(0) + o\left(\frac{1}{nb_n}\right) = o(1), \text{ by } (C3),$$

and

$$\frac{A_2}{\left(\mathbb{E}\left[\hat{\vartheta}_n(0)\right]\right)^2} = \left(\frac{1}{(f_X(0))^2} + o(1)\right) O\left(\frac{1}{nb_n}\right) = O\left(\frac{1}{nb_n}\right).$$

The last two equalities, imply that

$$\mathbb{E}\left[\hat{r}_n(0)\right] = \frac{\mathbb{E}\left[\hat{g}_n(0)\right]}{\mathbb{E}\left[\hat{\vartheta}_n(0)\right]} + O\left(\frac{1}{nb_n}\right).$$

Now combining $(C1)$, $(C3)$ and $(C4)$, we obtain

$$\frac{b_n^2}{2}\int u^2\Big[g''(\lambda u b_n) - g''(0)\Big]\psi(u)\,du = o(1). \tag{15}$$

Moreover, combining (7) and (15) we can write

$$\mathbb{E}\left[\hat{g}_n(0)\right] = g(0) + \frac{b_n^2}{2} g''(0)\int u^2\psi(u)\,du,$$

whence

$$\mathbb{E}[\hat{\vartheta}_n(0)] = f_X(0) + \frac{b_n^2}{2} f_X''(0)\int u^2\psi(u)du.$$

Then

$$\mathbb{E}\left[\hat{r}_n(0)\right] = \frac{g(0) + \frac{b_n^2}{2} g''(0)\int u^2\psi(u)\,du}{f_X(0) + \frac{b_n^2}{2} f_X''(0)\int u^2\psi(u)\,du} + O\left(\frac{1}{nb_n}\right) = H_n(0) + O\left(\frac{1}{nb_n}\right).$$

Next, we will obtain an equivalent expression for $H_n(0)$. Taking the conjugate, we have

$$\begin{aligned} H_n(0) \quad = \quad & \Bigg[g(0)f_X(0) + \frac{b_n^2\int u^2\psi(u)\,du}{2}\Big(g''(0)f_X(0) - f_X''(0)g(0)\Big) - \left(\frac{b_n^2}{2}\right)^2 f_X''(0)g''(0) \\ & \left(\int u^2\psi(u)\,du\right)^2\Bigg]\left[D_n(0)\right]^{-1}, \end{aligned}$$

where $D_n(0) = (f_X(0))^2 - \left(\frac{b_n^2 f_X''(0) \int u^2 \psi(u)\, du}{2}\right)^2$. From $(C2)$ and $(C3)$, we obtain $[D_n(0)]^{-1} = (f_X(0))^{-2} + o(1)$. Thus,

$$
\begin{aligned}
H_n(0) &= \left[g(0) f_X(0) + \frac{b_n^2}{2} \left[g''(0) f_X(0) - f_X''(0) g(0) \right] \int u^2 \psi(u)\, du \right.\\
&- \left. \left(\frac{b_n^2}{2} \right)^2 f_X''(0) g''(0) \left(\int u^2 \psi(u)\, du \right)^2 \right] \left[(f_X(0))^{-2} + o(1) \right]\\
&= r(0) + \frac{b_n^2}{2} \left[\frac{g''(0) f_X(0) - f_X''(0) g(0)}{(f_X(0))^2} \right] \int u^2 \psi(u)\, du + o\left(b_n^2\right).
\end{aligned}
$$

Then

$$
\mathbb{E}\left[\hat{r}_n(0)\right] = r(0) + \frac{b_n^2}{2} \left[r''(0) + 2\, \frac{f_X'(0) r'(0)}{f_X(0)} \right] \int u^2 \psi(u)\, du + o(b_n^2) + O\left(\frac{1}{n b_n}\right).
$$

By $(C3)$, we have $(n b_n)^{-1} = o(1)$. The result now follows combining the last two equalities. □

Theorem 4. *Under the conditions* $(C1) - (C5)$, *we have*

$$
\begin{aligned}
\mathbb{E}\left[\hat{r}_n(x)\right] &= r(0) + r'(0)\, b_n \int_{-c}^{B} u\, \psi(u)\, du + \frac{b_n^2}{2} \left[r''(0) + 2 \frac{r'(0) f_X'(0)}{f_X(0)} \right] \int_{-c}^{B} u^2\, \psi(u)\, du\\
&- \frac{g'(0) f_X'(0)\, b_n^2}{\left[f_X(0)\right]^2} \left[\int_{-c}^{B} u\, \psi(u)\, du \right]^2 + o\left(b_n^2\right).
\end{aligned}
\tag{16}
$$

Proof. Replacing z with x into (12) and taking the expectation, we obtain

$$
\mathbb{E}\left[\hat{r}_n(x)\right] = \frac{\mathbb{E}\left[\hat{g}_n(x)\right]}{\mathbb{E}\left[\hat{\vartheta}_n(x)\right]} - \frac{\dot{A}_1}{\left[\mathbb{E}\left[\hat{\vartheta}_n(x)\right]\right]^2} + \frac{\dot{A}_2}{\left[\mathbb{E}\left[\hat{\vartheta}_n(x)\right]\right]^2},
$$

where

$$
\dot{A}_1 = \mathbb{E}\left[\left(\hat{\vartheta}_n(x) - \mathbb{E}\left[\hat{\vartheta}_n(x)\right] \right) \hat{g}_n(x) \right] \quad \text{and} \quad \dot{A}_2 = \mathbb{E}\left[\left(\hat{\vartheta}_n(x) - \mathbb{E}\left[\hat{\vartheta}_n(x)\right] \right)^2 \hat{r}_n(x) \right].
$$

Combining (9) and (10), both hold true under the hypotheses of Theorem 4, we can write

$$
\mathbb{E}\left[\hat{g}_n(x)\right] = \int_{-c}^{B} \left[g(0) + b_n g'(0) u + \frac{(b_n u)^2}{2} \left(g''(0) + \left[g''(x + \lambda u b_n) - g''(0) \right] \right) \right] \psi(u) du. \tag{17}
$$

Remember that, $(C3)$ allows us to suppose that $b_n < 1$. Now, combining $(C1)$, $(C3)$, $(C4)$ with (17), we have that $\mathbb{E}[\hat{g}_n(x)] = g(0) \int_{-c}^{B} \psi(u)\, du + o(1)$, whence $\mathbb{E}[\hat{\vartheta}_n(x)] = f_X(0) \int_{-c}^{B} \psi(u)\, du +$

$o(1)$. Moreover, combining the fact that $((X_i, Y_i), V_i)$, $1 \leq i \leq n$, are identically distributed, with the previous equalities and $(C3)$, we have

$$\begin{aligned}\dot{A}_1 &= \frac{1}{(na_n)^2}\sum_{i=1}^{n} Cov\left[Y_i U_i, U_i\right] = \frac{1}{na_n}\mathbb{E}\big[\hat{g}_n(x)\big] - \frac{1}{n}\mathbb{E}\big[\hat{g}_n(x)\big]\,\mathbb{E}\big[\hat{\vartheta}_n(x)\big] \\ &= \frac{1}{na_n}\, g(0)\int_{-c}^{B}\psi(u)\,du + o\left(\frac{1}{nb_n}\right),\end{aligned}$$

where in this case $U = \mathbb{1}_{\left[0,\varphi\left(\frac{X-x}{b_n}\right)\right]}(V) = U^2$. On the other hand, by $(C5)$ there exists $C > 0$ such that $|\hat{r}_n(0)| \leq C$. Thus, we can write

$$|\dot{A}_2| \leq C\mathbb{E}\left[\left[\hat{\vartheta}_n(x) - \mathbb{E}[\hat{\vartheta}_n(x)]\right]^2\right] = \frac{C}{na_n^2}\left(\mathbb{E}\left[U^2\right] - \left(\mathbb{E}\left[U\right]\right)^2\right) = \frac{C}{na_n}\mathbb{E}\left[\frac{U}{a_n}\right](1 - \mathbb{E}\left[U\right]).$$

Now setting $Y = 1$ and combining with (17), we can write

$$\mathbb{E}[\hat{\vartheta}_n(x)] = \int_{-c}^{B}\left[f_X(0) + b_n f_X'(0)u + \frac{b_n^2}{2}u^2 f_X''(x + \lambda u b_n)\right]\psi(u)\,du.$$

Moreover, $(C1)$ implies that f_X'' is bounded in the neighborhood of x, and $(C3)$ allows us to suppose that $b_n < 1$. Thus, we can bound $\mathbb{E}\left[\hat{\vartheta}_n(x)\right]$. Besides, by (2) we can bound $(1 - \mathbb{E}[U])$. Then $\dot{A}_2 = O(\frac{1}{nb_n})$. Therefore $\frac{\dot{A}_1}{(\mathbb{E}[\hat{\vartheta}_n(x)])^2} = o(1)$ and $\frac{\dot{A}_2}{(\mathbb{E}[\hat{\vartheta}_n(x)])^2} = O(\frac{1}{nb_n})$. In consequence,

$$\mathbb{E}\left[\hat{r}_n(x)\right] = \frac{\mathbb{E}\left[\hat{g}_n(x)\right]}{\mathbb{E}\left[\hat{\vartheta}_n(x)\right]} + O\left(\frac{1}{na_n}\right).$$

Now combining $(C1)$, $(C3)$ and $(C4)$, we obtain

$$\frac{b_n^2}{2}\int_{-c}^{B} u^2\Big[g''(x + \lambda u b_n) - g''(0)\Big]\psi(u)\,du = o(1).$$

The previous equality allows us to rewrite (17) as

$$\mathbb{E}[\hat{g}_n(x)] = \int_{-c}^{B}[g(0) + b_n g'(0)u + \frac{b_n^2}{2}g''(0)u^2]\psi(u)\,du,$$

whence

$$\mathbb{E}[\hat{\vartheta}_n(x)] = \int_{-c}^{B}[f_X(0) + b_n\, f_X'(0)u + \frac{b_n^2}{2}f_X''(0)]\psi(u)\,du.$$

Then

$$\mathbb{E}\left[\hat{r}_n(x)\right] = \frac{g(0)C_c + C_{n,g}(0)}{f_X(0)C_c + C_{n,f_X}(0)} + O\left(\frac{1}{nb_n}\right) = \dot{H}_n(0) + O\left(\frac{1}{nb_n}\right), \tag{18}$$

where

$$C_c = \int_{-c}^{B}\psi(u)\,du$$

and

$$C_{n,q}(0) = b_n q'(0) \int_{-c}^{B} u\,\psi(u)\,du + \frac{b_n^2}{2} q''(0) \int_{-c}^{B} u^2 \psi(u)\,du, \text{ for } q = g, f_X.$$

Next, an equivalent expression for $\dot{H}_n(0)$ will be obtained. Taking the conjugate and combining the following equalities

$$\begin{array}{ll}
(i) & C_{n,f_X}(0) = o(1), \text{ by } (C3), \\
(ii) & \dfrac{1}{(f_X(0)\,C_c)^2 - \left(C_{n,f_X}(0)\right)^2} = \dfrac{1}{(f_X(0)\,C_c)^2} + o(1), \text{ by } (i), \\
(iii) & C_{n,g}(0) C_{n,f_X}(0) = b_n^2 g'(0) f_X'(0) \left[\displaystyle\int_{-c}^{B} u\psi(u)du\right]^2 + ø\left(b_n^2\right), \text{ by} (C3), \\
(iv) & g(0) f_X(0) = r(0)\left[f_X(0)\right]^2, \text{ since } g(x) = r(x) f_X(x), \\
(v) & f_X(0) g'(0) - g(0) f_X'(0) = r'(0)\left[f_X(0)\right]^2, \text{ given that } r(x) = \dfrac{g(x)}{f_X(x)}, \\
(vi) & g''(0) f_X(0) - f_X''(0) g(0) = r''(0)\left[f_X(0)\right]^2 + 2 f_X(0) r'(0) f_X'(0),
\end{array}$$

we can write

$$\begin{aligned}
\dot{H}_n(0) &= r(0) + r'(0)\,b_n \int_{-c}^{B} u\,\psi(u)\,du + b_n^2 \left[r''(0) + 2\frac{r'(0) f_X'(0)}{f_X(0)}\right] \int_{-c}^{B} u^2 \psi(u)\,du \\
&+ \frac{g'(0) f_X'(0)\, b_n^2}{\left[f_X(0)\right]^2} \left[\int_{-c}^{B} u\,\psi(u)\,du\right]^2 + o\left(b_n^2\right).
\end{aligned}$$

By $(C3)$, we have that $(nb_n)^{-1} = o(1)$. The result now follows combining (18) and the last two equalities. □

Next, the results that will allow us to eliminate the boundary effects in estimator (3) are presented, which were used in [8], Theorems 3 and 4, to eliminate the boundary effects in estimator (1). The following theorem in particular will allow us to control suitably the constants that define the bias of estimator (3) and it justifies a condition in the criterion introduced to eliminate the terms with coefficients b_n in all the expressions above.

Theorem 5. *Under condition $(C4)$, we have that for $M > 0$ there exists $B' > 0$ such that*

$$v = \int_{-B'}^{B'} u^2 \psi(u)\,du \ \leq\ M.$$

Proof. Similar to the proof of Theorem 3 in [8]. □

Observe that combining $(C4)$ and Theorem 5 we obtain

$$v = \int_{-B'}^{B'} u^2 \psi(u)\,du = \int u^2 \psi(u)\,du \leq M,$$

for $B' > B$. Now, we can redefine ψ as

$$\psi(u) = \frac{\varphi(u)}{\int \varphi(u)\, du}\, \mathbb{1}_{[-B',B']}(u), \quad B' \leq B. \tag{19}$$

To give a simple and effective solution to the boundary problem, without boundary corrections, a criterion will be implemented which will allow us to eliminate the terms with coefficients b_n in all above expressions, making $\int_{-c}^{B} u\,\psi(u)\,du = 0$ for each $c \in (0, b_n]$. Such criterion is based on deriving a thinning function φ as the solution to the following variational problem:

$$\begin{aligned}
\text{Maximizing}: &\int \varphi(u)\,du.\\
\text{Subject to}: &\int \varphi^2(u)\,du \;=\; k,\\
&\int u\,\varphi(u)\,du \;=\; 0,\\
&\int \left(u^2 - v\right)\,\varphi(u)\,du \;=\; 0,
\end{aligned} \tag{VP}$$

$$k > 0,\ \varphi(u) = 0 \text{ for } u \in (-B,B)^c, \varphi(0) > 0,\ \varphi(u) \in [0,1],\ v \in (0, M].$$

Theorem 6. *The solution of (VP) is given by*

$$\varphi_k(u) = \left[1 - \left(\frac{16}{15\,k}\right)^2 u^2\right] \mathbb{1}_{\left[-\frac{15}{16}k,\frac{15}{16}k\right]}(u),\ k > 0. \tag{20}$$

In particular, for each $c \in (0, b_n]$ we have

$$\varphi_c(u) = \left[1 - \left(\frac{16}{15\,c}\right)^2 u^2\right] \mathbb{1}_{\left[-\frac{15}{16}c,\frac{15}{16}c\right]}(u). \tag{21}$$

Proof. Similar to the proof of Theorem 4 in [8]. □

From (16) w.t.f. φ_c we obtain

$$\mathbb{E}\left[\hat{r}_n(x)\right] = r(0) + \frac{b_n^2}{2}\left[r''(0) + 2\frac{r'(0)f_X'(0)}{f_X(0)}\right]\int_{-B'}^{B'} u^2\psi_c(u)\,du + o\left(b_n^2\right), \tag{22}$$

where $0 < c \leq 1$, $0 < B' \leq \frac{15}{16}c$, ψ_c is given by (19) w.t.f. φ_c, and φ_c is given by (21). Thus, estimator (3) does not present the boundary effects at x when the thinning function is φ_c. Moreover, combining Theorem 3 with Theorem 3.1 in [5], we obtain

$$\mathbb{E}\left[\hat{r}_n(z)\right] = r(z) + \frac{b_n^2}{2}\left[r''(z) + 2\frac{r'(z)f_X'(z)}{f_X(z)}\right]\int_{-B'}^{B'} u^2\psi_k(u)\,du + o\left(b_n^2\right),$$

for each $k > 1$ and $z \in \{0\} \cup (b_n, \infty)$, where $B' \leq \frac{15}{16}k$, ψ_k is given by (19) w.t.f. φ_k, and φ_k is given by (20). Thus, estimator (3) does not present the boundary effects at $z \in \{0\} \cup (b_n, \infty)$ when the thinning function is φ_k, for each $k > 1$.

On the other hand, denoting the Epanechnikov kernel by K_E, and replacing k with $\frac{5}{3}$ and M with $M_E = \int u^2 K_E(u)\,du$ into (VP), we have that estimator (3) w.t.f. $\varphi_{\frac{5}{3}}$, is the estimator studied in [5]. That is, the estimator introduced in [5] is a particular case of the class of estimators defined by (3) w.t.f. φ_k, for each $k > 1$. Moreover, the results obtained in [5] allow to guarantee that φ_k minimize $MSE[\hat{r}_n(z)]$, for each $k > 1$ and $z \in \{0\} \cup (b_n, \infty)$. Particularly in [5], it was shown that $MSE[\hat{r}_n(t)] \leq MSE\left[\hat{r}_{NW_n}(t)\right]$ for each $t \in \mathbb{R}$, where $\hat{r}_n$ is given by (3) w.t.f. $\varphi_{\frac{5}{3}}$ and $\hat{r}_{NW_n}$ is the classical kernel regression estimator.

Next, the boundary estimator introduced in [8] and the main result of the above article are presented. The inclusion of the above results is based on the following two reasons. First, because the boundary regression estimator will be defined in terms of the estimator proposed in [8]. Second, to highlight two asymptotic properties of the boundary estimator defined in [8].

Definition 3. Let $X_1, \ldots, X_n$ be an independent random sample of X. Let $V_1, \ldots, V_n$ be independent random variables uniformly distributed on $[0, 1]$ and independent of $X_1, \ldots, X_n$. Then, the boundary fuzzy set estimator of the density function f_X at the point $x \in (0, b_n]$ is defined as

$$\tilde{\vartheta}_n(x) = \frac{1}{n\,a_n} \sum_{i=1}^{n} \mathbb{1}_{\left[0,\, \varphi_c\left(\frac{X_i - x}{b_n}\right)\right]}(V_i) = \frac{1}{na_n}\,\tau_n^{(c)}(x), \tag{23}$$

where $0 < c \leq 1$, $a_n = b_n \int \varphi_c(u)\,du$, and φ_c is given by (21).

Theorem 7. *Under conditions (C1)-(C3), we have*

$$\mathbb{E}\left[\tilde{\vartheta}_n(x) - f_X(0)\right] = \frac{b_n^2}{2} f_X''(0) \int_{-B'}^{B'} u^2 \psi_c(u)\,du + o\left(b_n^2\right),$$

and

$$Var\left[\tilde{\vartheta}_n(x)\right] = \frac{1}{n\,b_n \int \varphi_c(u)\,du}\, f_X(0) + o\left(\frac{1}{n\,b_n}\right),$$

where $0 < c \leq 1$, $0 < B' \leq \frac{15}{16}c$, ψ_c *is given by (19) w.t.f.* φ_c, *and* φ_c *is given by (21).*

Next, the boundary fuzzy set estimator of r and the main result of this paper are presented.

Definition 4. Let $((X_1, Y_1), V_1), \ldots, ((X_n, Y_n), V_n)$ be independent copies of a random vector $((X, Y), V)$, where $V_1, \ldots, V_n$ are independent random variables uniformly distributed on $[0, 1]$, and independent of $(X_1, Y_1), \ldots, (X_n, Y_n)$. Then, the boundary fuzzy set estimator of the regression function r at the point $x \in (0, b_n]$ is defined as

$$\tilde{r}_n(x) = \begin{cases} \dfrac{\sum_{i=1}^{n} Y_i\, \mathbb{1}_{\left[0, \varphi_c\left(\frac{X_i - x}{b_n}\right)\right]}(V_i)}{\tau_n^{(c)}(x)} & \text{if} \quad \tau_n^{(c)}(x) \neq 0 \\ 0 & \text{if} \quad \tau_n^{(c)}(x) = 0, \end{cases} \tag{24}$$

where $0 < c \leq 1$, $0 < B' \leq \frac{15}{16}\,c$, $\tau_n^{(c)}$ is given in (23), and φ_c is given by (21).

Lemma 1. *Under the conditions* $(C1)-(C6)$, *we have*

$$\mathbb{E}\left[\tilde{r}_n(x)-r(0)\right]=\frac{b_n^2}{2}\left[r''(0)+2\,\frac{f_X'(0)\,r'(0)}{f_X(0)}\right]\int_{-B'}^{B'}u^2\psi_c(u)du+o\left(b_n^2\right)$$

and

$$Var\left[\tilde{r}_n(x)\right]=\frac{1}{n\,b_n\int\varphi_c(u)\,du}\left[\frac{\phi(0)-r^2(0)}{f_X(0)}\right]+o\left(\frac{1}{n\,b_n}\right),$$

where $0<c\leq 1$, $0<B'\leq\frac{15}{16}c$, ψ_c *is given by (19) w.t.f.* φ_c, *and* φ_c *is given by (21).*

Proof. From (3) w.t.f. φ_c, we have $\tilde{r}_n(x)=\hat{r}_n(x)$. Now, combining the above equality with (22), we obtain the expression for $\mathbb{E}\left[\tilde{r}_n(x)-r(0)\right]$. For the proof of $Var\left[\tilde{r}_n(x)\right]$, follow [5] considering φ_c as thinning function, keeping in mind that $q(x)=q(0)+o(1)$, for $q=f_X, r, \phi$. □

For $z\geq b_n$, $\tilde{r}_n(z)$ reduces to the fuzzy set regression estimator $\hat{r}_n(z)$ given by (3) w.t.f. φ_1. Thus, $\tilde{r}_n(z)$ is a natural boundary continuation of estimator (3) w.t.f. φ_1. Moreover, the results obtained in [5] allow us to guarantee that the thinning function φ_c locally minimizes $MSE[\tilde{r}_n(x)]$, for each $c\in(0,1]$.

Next, asymptotic formulae for the optimal smoothing parameter and optimal mean square error of (24), b_n^* and MSE^*, are presented, which are an immediate consequence of Lemma 1, and they guarantee that estimator (24) is locally optimal in terms of rates of convergence (see [26]).

Corollary 3. *Under the conditions* $(C1)-(C6)$, *we have*

$$b_n^*=n^{-1/5}\left[\frac{\left(\int\varphi_c(u)\,du\right)^{-1}\left(\frac{\phi(0)-r^2(0)}{f_X(0)}\right)}{\left(r''(0)+2\,\frac{f_X'(0)\,r'(0)}{f_X(0)}\right)^2\left(\int_{-B'}^{B'}u^2\psi_c(u)du\right)^2}\right]^{1/5}$$

and

$$MSE^*\left[\tilde{r}_n(\dot{x})\right]=\frac{5}{4}\,n^{-4/5}\left[\frac{\left(\frac{\phi(0)-r^2(0)}{f_X(0)}\right)^4\left(r''(0)+2\,\frac{f_X'(0)\,r'(0)}{f_X(0)}\right)^2}{\left(\int_{-B'}^{B'}u^2\psi_c(u)du\right)^{-2}\left(\int\varphi_c(u)\,du\right)^4}\right]^{1/5},$$

where $0<c\leq 1$, $\dot{x}=c\,b_n^*$, $0<B'\leq\frac{15}{16}c$, ψ_c *is given by (19) w.t.f.* φ_c, *and* φ_c *is given by (21).*

3 Simulations results

In this section some simulation results are presented, which are only designed to illustrate the performance of (24) at points near 0 in a b_n spread neighborhood. For purposes of comparison, the estimator introduced in [24] was also considered. It is important to remark here that, the particular choice above was based mainly on the results of the simulations obtained in [24], for the two regression models and two densities considered in this section, which showed that the boundary kernel regression estimator performed quite well when it was compared with the local linear and $\hat{r}_{NW_n}$ estimators. Among other reasons that sustained the above particular choice, the theoretical properties that are shared by the boundary kernel regression estimator defined in [24] and the proposed boundary fuzzy set regression estimator are highlighted: non-negativity, "natural boundary continuation" and they improve the bias but holding on to the low variances. Properties not justified will be verified in this section. The author considers that the previous comments and discussion motivated by the literature review results, justify considering only the estimator given by [24] to develop the simulations.

The general boundary kernel estimator introduced in [24] is defined as

$$\tilde{r}_{K_n}(l) = \frac{\sum_{i=1}^{n} Y_i \left\{ K\left(\frac{l+w_1(X_i)}{h_n}\right) + K\left(\frac{l-w_1(X_i)}{h_n}\right)\right\}}{\sum_{i=1}^{n} \left\{ K\left(\frac{l+w_2(X_i)}{h_n}\right) + K\left(\frac{l-w_2(X_i)}{h_n}\right)\right\}}, \tag{25}$$

where $l = s\,h_n$, $0 \leq s \leq 1$, h_n is the smoothing parameter and K is a kernel function of order 2. Moreover, w_k is a transformation defined as

$$w_k(y) = y + \frac{1}{2} d_k\, y^2 + \lambda_0\, (d_k)^2\, y^3,\ \ k = 1, 2,$$

where

$$d_1 = w_1''(0) = \frac{g'(0)}{g(0)}\, D_{K,s}, \quad d_2 = w_2''(0) = \frac{f_X'(0)}{f_X(0)}\, D_{K,s},$$

and λ_0 is a constant such that $12\lambda_0 > 1$, with

$$D_{K,s} = \frac{2\int_s^1 (u-s)K(u)\,du}{2\int_s^1 (u-s)K(u)\,du + s}.$$

To assess the effect of the boundary estimators at points near 0 in a b_n spread neighborhood, the following models are studied:

$$\text{Model 1}: r_1(z) = 2\,z + 1 \quad \text{and} \quad \text{Model 2}: r_2(z) = 2\,z^2 + 3\,z + 1,$$

where $z \in [0, \infty)$ and errors ε_j, are assumed to be standard normally distributed independent random variables. Likewise, consider two cases of density f_X with support $[0, \infty)$:

$$\text{Density 1}: f_{1_X}(z) = \exp(-z) \quad \text{and} \quad \text{Density 2}: f_{2_X}(z) = \frac{2}{\pi(1+z^2)}.$$

It is important to emphasize that, for $z \geq h_n$ the estimator (25) reduces to the classical kernel estimator $\hat{r}_{NW_n}$. Thus, (25) is a natural boundary continuation of $\hat{r}_{NW_n}$ (see [24], Section 2).

On the other hand, the following optimal global smoothing parameters were implemented as the smoothing parameters of both (24) and (25) estimators (see Theorems 3.1 and 2.4.1 in [5] and [12], respectively):

$$b_n = \left[\frac{C}{n}\, \frac{\left(\int \varphi_{\frac{5}{3}}(u)\,du\right)^{-1}}{\left(\int u^2 \psi_{\frac{5}{3}}(u)\,du\right)^2} \right]^{1/5} \tag{26}$$

and

$$h_n = \left[\frac{C}{n} \frac{\int K_E^2(u)du}{\left(\int u^2 K_E(u)du\right)^2} \right]^{1/5}, \tag{27}$$

where

$$C = \frac{\int \left(\frac{\phi(u) - r^2(u)}{f_X(u)} \right) du}{\int \left(r''(u) + 2\, \frac{f_X'(u)\, r'(u)}{f_X(u)} \right)^2 du}, \tag{28}$$

$\varphi_{\frac{5}{3}}$ is given by (20), $\psi_{\frac{5}{3}}$ is given by (19) w.t.f. $\varphi_{\frac{5}{3}}$, and $v = \int u^2 \psi_{\frac{5}{3}}(u)du \leq \int u^2 K_E(u)\, du = M_E$ (see [5], Section 3).

It is important to emphasize that the estimator (3) w.t.f. $\varphi_{\frac{5}{3}}$ has better performance than the estimator $\hat{r}_{NW_n}$ (see [5]). Moreover, the reason for using an optimal global smoothing parameter as the smoothing parameter is that the comparisons based on the optimal smoothing parameter are more convincing than comparisons based on approximated smoothing parameters, which—because of the quality or otherwise of the smoothing parameter selection method—might be misleading. Also, a global rather than local smoothing parameter choice is made, because this is much more likely to be used in applications.

The following formulas and values are used in all simulations, $K_E(u) = \frac{3}{4}(1 - x^2)\mathbb{1}_{[-1,1]}(u)$, $\lambda_0 = 0.0933$, $v = M_E/4$, $s = 0.1, 0.2, 0.3, 0.4, 0.5$ and $c = (h_n/b_n)\, s$, where the smoothing parameters b_n and h_n are given by (26) and (27), respectively. Moreover, the simulated sample sizes are $n = 50$ and $n = 500$, and all results are calculated by averaging over 1000 trials. Simultaneously, for each regression model and each density, the absolute bias (absolute value of the estimated value minus the true value) and the MSE of both estimators are calculated, at the points $x = c\, b_n$ and $l = s\, h_n$ of the following boundary regions $(0, b_n)$ and $(0, h_n)$. Nonetheless, the comparison of both (24) and (25) estimators through the mean integrated squared error is not convenient, since to the two regression models and two densities considered the boundary regions satisfy the following property $(0, h_n] \subset (0, b_n]$. The results are shown in Tables 3 and 3. Furthermore, a close examination of Tables 3 and 3 allows us to see that for the two regression models and two densities considered the proposed boundary estimator is the best in terms of MSE at points near 0 in a b_n spread neighborhood, since it has low bias and extremely low variance, guaranteeing that $MSE\,[\tilde{r}_n(x)] \leq MSE[\tilde{r}_{K_n}(x)]$ at the points x near 0 in a b_n spread neighborhood. Additionally, these properties are apparently preserved over the rest of the boundary regions. Thus, the proposed boundary estimator outperforms the estimator introduced in [24].

Table 1: Bias and MSE of the indicated regression estimators at boundary, case of density 1.

		$\tilde{r}_n$			Model 1			$\tilde{r}_{K_n}$		
		$b_n = 3.5045$			$n = 50$			$h_n = 2.1047$		
		c			—			s		
0.0601	0.1201	0.1802	0.2402	0.3003	—	0.1000	0.2000	0.3000	0.4000	0.5000
		| Bias |			—			| Bias |		
0.0011	0.0560	0.1332	0.2454	0.3812	—	0.1726	0.4102	0.5823	0.7552	0.9440
		MSE			—			MSE		
0.0000	0.0031	0.0177	0.0602	0.1453	—	0.0298	0.1682	0.3391	0.5703	0.8911
		$b_n = 2.2112$			$n = 500$			$h_n = 1.3280$		
		c			—			s		
0.0601	0.1201	0.1802	0.2402	0.3003	—	0.1000	0.2000	0.3000	0.4000	0.5000
		| Bias |			—			| Bias |		
0.0101	0.0240	0.0528	0.0963	0.1515	—	0.2347	0.2842	0.3228	0.3807	0.4496
		MSE			—			MSE		
0.0001	0.0006	0.0028	0.0093	0.0230	—	0.0551	0.0807	0.1042	0.1449	0.2022
		$\tilde{r}_n$			Model 2			$\tilde{r}_{K_n}$		
		$b_n = 1.2978$			$n = 50$			$h_n = 0.7794$		
		c			—			s		
0.0601	0.1201	0.1802	0.2402	0.3003	—	0.1000	0.2000	0.3000	0.4000	0.5000
		| Bias |			—			| Bias |		
0.0383	0.0200	0.0225	0.0408	0.0579	—	0.1667	0.2058	0.2433	0.2762	0.3045
		MSE			—			MSE		
0.0015	0.0004	0.0005	0.0017	0.0034	—	0.0278	0.0423	0.0592	0.0763	0.0927
		$b_n = 0.8189$			$n = 500$			$h_n = 0.4918$		
		c			—			s		
0.0601	0.1201	0.1802	0.2402	0.3003	—	0.1000	0.2000	0.3000	0.4000	0.5000
		| Bias |			—			| Bias |		
0.0036	0.0012	0.0062	0.0111	0.0230	—	0.0727	0.0907	0.1047	0.1133	0.1162
		MSE			—			MSE		
0.0000	0.0000	0.0000	0.0001	0.0005	—	0.0053	0.0082	0.0110	0.0128	0.0135

Table 2: Bias and MSE of the indicated regression estimators at boundary, case of density 2.

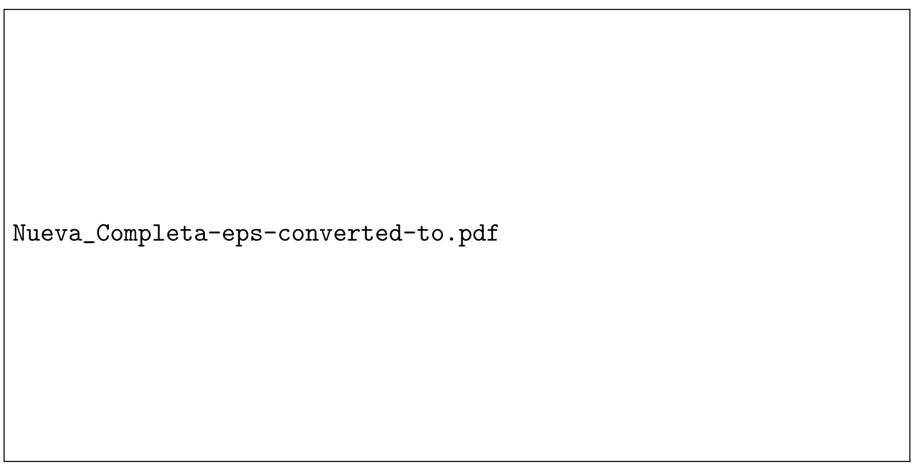
		$\tilde{r}_n$			Model 1			$\tilde{r}_{K_n}$		
		$b_n = 2.1614$			n = 50			$h_n = 1.2980$		
		c			—			s		
0.0601	0.1201	0.1802	0.2402	0.3003	—	0.1000	0.2000	0.3000	0.4000	0.5000
		| Bias |			—			| Bias |		
0.0201	0.0148	0.0359	0.0734	0.1233	—	0.0996	0.2173	0.3078	0.3800	0.4398
		MSE			—			MSE		
0.0004	0.0002	0.0013	0.0054	0.0152	—	0.0099	0.0472	0.0947	0.1444	0.1934
		$b_n = 1.3637$			n = 500			$h_n = 0.8190$		
		c			—			s		
0.0601	0.1201	0.1802	0.2402	0.3003	—	0.1000	0.2000	0.3000	0.4000	0.5000
		| Bias |			—			| Bias |		
0.0050	0.0017	0.0042	0.0179	0.0378	—	0.0652	0.1161	0.1502	0.1702	0.1801
		MSE			—			MSE		
0.0000	0.0000	0.0000	0.0003	0.0014	—	0.0042	0.0135	0.0226	0.0290	0.0324
		$\tilde{r}_n$			Model 2			$\tilde{r}_{K_n}$		
		$b_n = 1.0466$			n = 50			$h_n = 0.6285$		
		c			—			s		
0.0601	0.1201	0.1802	0.2402	0.3003	—	0.1000	0.2000	0.3000	0.4000	0.5000
		| Bias |			—			| Bias |		
0.0710	0.0309	0.0284	0.0088	0.0132	—	0.0720	0.1305	0.1670	0.1806	0.1708
		MSE			—			MSE		
0.0050	0.0010	0.0008	0.0001	0.0002	—	0.0052	0.0170	0.0279	0.0326	0.0292
		$b_n = 0.6603$			n = 500			$h_n = 0.3966$		
		c			—			s		
0.0601	0.1201	0.1802	0.2402	0.3003	—	0.1000	0.2000	0.3000	0.4000	0.5000
		| Bias |			—			| Bias |		
0.0064	0.0040	0.0013	0.0028	0.0088	—	0.0333	0.0579	0.0697	0.0678	0.0544
		MSE			—			MSE		
0.0000	0.0000	0.0000	0.0000	0.0001	—	0.0011	0.0033	0.0049	0.0046	0.0030

4 Data analysis

The proposed estimator was tested over one well-known real dataset to demonstrate its usefulness in practical applications. For purposes of comparison, the particular classical kernel regression estimator introduced in [22] was considered. With respect to this last point, a brief discussion will be developed at the end of this section. The real dataset is the Motorcycle data found in Appendix 2-Table 1 of [14], where the experiment, a simulated motorcycle crash, is described in detail in [23]. It has $n = 133$ observations of X, where the X-values denote time (in milliseconds) after a simulated impact with motorcycles and the response variable Y is the head acceleration (in g) of a post mortem human test object. To facilitate the comparison of the effectiveness of estimators (24), $\hat{r}_{NW_n}$, (3) w.t.f. $\varphi_{\frac{5}{3}}$, and (3) w.t.f. φ_1, the performance of each estimator will be graphed over the dataset through an average of 1000 trials. Moreover, for the fixed random sample, $X_1, \ldots, X_n$, a sample of independent random variables $V_1^{(d)}, \ldots, V_n^{(d)}$ will be used, where $V_i^{(d)}$ is uniformly distributed on $[0, 1]$ for each i, d, $1 \leq i \leq n$ and $1 \leq d \leq 1000$.

In order to simulate the performance of the estimator $\hat{r}_{NW_n}$, the idea in [22] will be adopted with smoothing parameter $h_n = 2.40$, and quartic kernel $K_Q(u) = (16/25)(1 - u^2)^2 \mathbb{1}_{[-1,1]}(u)$ instead of K_E. Simultaneously, for the fuzzy set estimation the smoothing parameter (26) will be implemented, with $v = M_Q/2$, where the approximate value of constant (28) was calculated through (27) using $h_n = 2.40$ and K_Q instead of K_E. Here $M_Q = \int u^2 K_Q(u)\, du$. Figures 4 and 4 show the simulation results together with the data points on $[0, 57.60]$ and $[0, 6.50]$, respectively. From Figures 4 and 4, the estimator (3) w.t.f. $\varphi_{\frac{5}{3}}$ does not appear to offer an appropriate approximation on large part of the region $[2.92, 57.60]$, when compared with the similar results produced by both (3) w.t.f. φ_1 and $\hat{r}_{NW_n}$ estimators. At the same time, Figure 4 shows that the presence of the boundary problem in (3) w.t.f. $\varphi_{\frac{5}{3}}$ was removed by (24), where both (24) and $\hat{r}_{NW_n}$ estimators fail miserably over their respective boundary regions, and only the estimator (24) follows the data very closely on the left side of the picture, by making the curve be constant, equal zero, over the region $[0, 1.305]$.

Finally, it is necessary to point out that [24] does not present a study of simulations with real data, and within the context of the above paper an appropriate criterion for the estimation of the parameters d_1 and d_2 was not proposed. The previously exposed explains the absence, in this section, of a study related to the graphical comparison of estimators (25) and (24). Remember that, within the proposed objectives in this paper only the estimation of the parameters associated with the fuzzy set estimation method will be considered. Consequently, the subjective choice of the values of the parameters d_1 and d_2 would be incorrect, since it could benefit the behavior of (24) and impair the behavior of (25), which would produce unreliable results.



$\tilde{r}_n$ -.-.-.-. $\hat{r}_{NW_n}$ $\hat{r}_n$ w.t.f. $\varphi_{\frac{5}{3}}$ $\hat{r}_n$ w.t.f. φ_1

Figure 1: Regression estimates to the motorcycle data at the points inside the region $[0, 57.60]$ using smoothing parameters $h_n = 2.40$ and $b_n = 2.92$, where $v = \frac{M_Q}{2}$. The circles indicate the raw data.

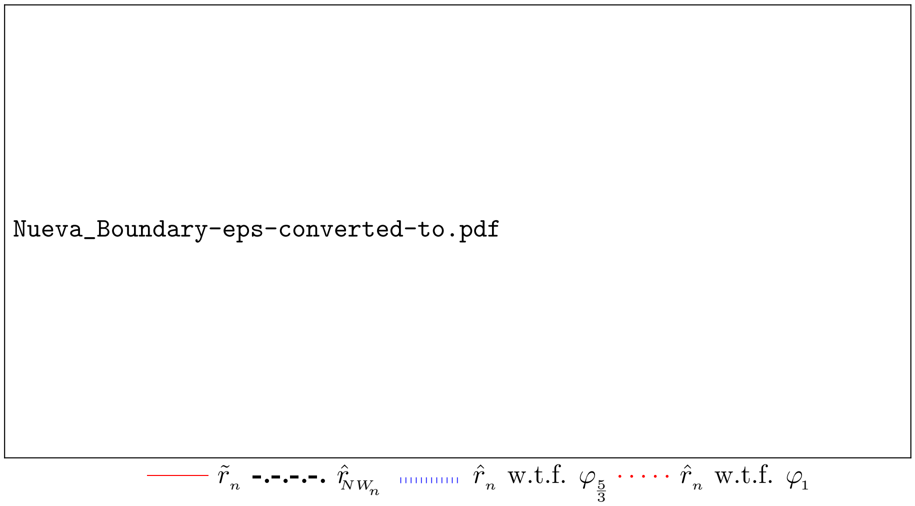


Figure 2: Regression estimates to the motorcycle data at the points inside the region $[0, 6.5]$ using smoothing parameters $h_n = 2.40$ and $b_n = 2.92$, where $v = \frac{M_Q}{2}$. The circles indicate the raw data.

5 Final remarks

In this paper, a new contribution in the area of regression estimation not based on kernels is presented, obtaining a natural extension of the results introduced in [8]. In particular, the boundary effects in the fuzzy set regression estimator are studied and removed, where it does not require boundary modifications to eliminate such effects, unlike most well-known kernel regression estimators. Moreover, among the desirable properties of the boundary fuzzy set estimator the non-negativity of the estimator is highlighted, as well as its performance which is generally very robust with respect to various shapes of regression and density functions, since it allows important reductions of the bias, while maintaining low variance. It is clear that no single existing estimator in the literature dominates all the others for all shapes of regression and density functions. Each estimator has certain advantages and works well at certain times. However, for the two regression models and the two density functions considered, the boundary fuzzy set regression estimator has better performance than the boundary regression estimator introduced in [24] at points near zero in a spread neighborhood of the smoothing parameter.

On the other hand, it is appropriate to highlight the important role that the thinning function plays in the results obtained, since its adequate construction allowed to eliminate the boundary effects in the fuzzy set regression estimator, giving each boundary point the least approximation between the boundary fuzzy set estimator and regression function. Finally, it is necessary to point out that through the thinning point process that describes the fuzzy set density estimation method, the set of observations considered can be characterized in a neighborhood of the estimation point, where the indicator functions that define the fuzzy set density estimator decide whether observation belongs on the neighborhood of the estimation point or not, and the thinning functions that define each indicator function are used to select the sample points with different probabilities, in contrast to the kernel estimators which assign equal weight to all points of the sample.

6 Acknowledgements

This research has been supported by a grant from the Academia de Ciencias de América Latina-ACAL.

References

[1] T. H. Ali, H. A. A.-M. Hayawi and D. S. I. Botani (2021). *Estimation of the bandwidth parameter in Nadaraya-Watson kernel non-parametric regression based on universal threshold level*, Commun. Stat. - Simul. Comput., 1–34. DOI: 10.1080/03610918.2021.1884719

[2] L. R. Cheruiyot (2020). *Local Linear Regression Estimator on the Boundary Correction in Nonparametric Regression Estimation*, JSTA, **19** (3), 460–471. DOI: 10.2991/jsta.d.201016.001

[3] L. R. Cheruiyot, O. R. Otieno and G. O. Orwa (2019). *A Boundary Corrected Non-Parametric Regression Estimator for Finite Population Total*,Int. J. Probab. Stat., **8** (3), 1–83. DOI: 10.5539/ijsp.v8n3p83

[4] F. Comte, and N. Marie (2021). *On a Nadaraya-Watson estimator with two bandwidths*, Electron. J. Statist, **15** (1), 2566–2607. DOI: 10.1214/21-EJS1849

[5] J. Fajardo (2012). *A Criterion for the Fuzzy Set Estimation of the Regression Function*, J. Prob. Stat., **2012**. DOI: 10.1155/2012/593036

[6] J. Fajardo (2014). *A criterion for the fuzzy set estimation of the density function*, Braz. J. Prob. Stat., **28** (3), 301–312. DOI: 10.1214/12-BJPS208

[7] J. Fajardo (2018). *Comparison of the Mean Square Errors of the Fuzzy Set Regression Estimator and Local Linear Regression Smoothers*, Int. J. Math. Stat., **19** (1), 74–89. http://www.ceser.in/ceserp/index.php/ijms/article/view/5355

[8] J. Fajardo and P. Harmath (2021). *Boundary estimation with the fuzzy set density estimator*, METRON, **79**, 285–302. DOI: 10.1007/s40300-021-00210-z

[9] J. Fajardo R. Ríos and L. Rodríguez (2010). *Properties of convergence of a fuzzy set estimator of the regression function*, J. Stat., Adv. Theory Appl., **3** (2), 79–112. Zbl 05800441

[10] J. Fajardo, R. Ríos and L. Rodríguez (2012). *Properties of convergence of a fuzzy set estimator of the density function*, Braz. J. Prob. Stat., **26** (2), 208–217. DOI: 10.1214/10-BJPS130

[11] M. Falk and F. Liese (1998). *Lan of thinned empirical processes with an application to fuzzy set density estimation*, Extremes, **1** (3), 323–349. DOI: 10.1023/A:1009981817526

[12] F. Ferraty, V. Núñez-Antón, and P. Vieu (2001). *Regresión No Paramétrica: Desde la Dimensión uno Hasta la Dimensión Infinita*, Servicio Editorial de la Universidad del Pais Vasco, Bilbao.

[13] P. Hall and B. U. Park (2002). *New methods for bias correction at endpoints and boundaries*, Ann. Statist., **30** (5), 1460–1479. DOI: 10.1214/aos/1035844983

[14] W. Härdle (1990). *Applied nonparametric regression*, Cambridge University Press.

[15] R. Hidayat, I. N. Budiantara, B. W. Otok and V. Ratnasari (2021). *The regression curve estimation by using mixed smoothing spline and kernel (MsS-K) model*, Commun. Stat. Theory Methods, **50** (17), 3942–3953. DOI: 10.1080/03610926.2019.1710201

[16] M. C. Jones and P. J. Foster (1996). *A simple nonnegative boundary correction method for kernel density estimation*, Stat. Sinica, **6** (4), 1005–1013. http://www3.stat.sinica.edu.tw/statistica/j6n4/j6n414/j6n414.htm

[17] R. J. Karunamuni and T. Alberts (2005). *On boundary correction in kernel density estimation*, Stat. Methodol., **2** (3), 191–212. DOI: 10.1016/j.stamet.2005.04.001

[18] B. C. Kouassi, O. Hili and E. Katchekpele (2021). *Nadaraya-Watson estimation of a nonparametric autoregressive mode*, MJM, **9** (4), 251–258. DOI: 10.26637/mjm904/009

[19] Y. Linke, I. Borisov, P. Ruzankin, V. Kutsenko, E. Yarovaya and S. Shalnova (2022). *Universal Local Linear Kernel Estimators in Nonparametric Regression*, Mathematics, **10** (15), 2693. DOI: 10.3390/math10152693

[20] E. A. Nadaraya (1964). *On estimating regression*, Theory Probab. Appl., **9**, 141–142. DOI: 10.1137/1109020

[21] R. D. Reiss (1993). *A Course on Point Processes*, Springer Series in Statistics, New York.

[22] I. Salgado-Ugarte, M. Shimizu and T. Taniuchi (1996). *Nonparametric regression: Kernel, WARP, and k-NN estimators*, S.T.B., **5**. https://EconPapers.repec.org/RePEc:tsj:stbull:y:1996:v:5:i:30:snp10

[23] G. Schmidt, R. Mattern and F. Schüler (1981). *Biomechanical investigation to determine physical and traumatological differentiation criteria for the maximum load capacity of head and vertebral column with and without protective helmet under effects of impact. EEC Research Program on Biomechanics of Impac*, Final Report Phase III, Project 65, Institut für Rechtsmedizin, Universität Heidelberg.

[24] K. Souraya, A. Sayah and Y. Djabrane (2015). *General method of boundary correction in kernel regression estimation*, Afr. Stat., **10**, 739–750. DOI: 10.16929/as/2015.729.66

[25] I. Sriliana, I . N. Budiantara and V. Ratnasari (2022). *A Truncated Spline and Local Linear Mixed Estimator in Nonparametric Regression for Longitudinal Data and Its Application*, Symmetry, **14** (12), 2687. DOI: 10.3390/sym14122687

[26] C. J. Stone (1980). *Optimal rates of convergence for nonparametric estimators*, Ann. Statist., **8**, 1348–1360. DOI: 10.1214/aos/1176345206

[27] G. S. Watson (1964). *Smooth regression analysis*, Sankhyā, Ser. *A*, **26**, 359–372. https://www.jstor.org/stable/25049340

[28] L. A. Zadeh (1965). *Fuzzy sets*, Inform. Control, **8**, 338–353.

[29] S. Zhang and R. J. Karunamuni (1998). *On kernel density estimation near endpoints*, J. Stat. Plan. Infer., **70** (2), 301–316. DOI: 10.1016/S0378-3758(97)00187-0

[30] S. Zhang, R. J. Karunamuni and M. C. Jones (1999). *An Improved Estimator of the Density Function at the Boundary*, J. Am. Stat. Assoc. **94** (448), 1231–1240. DOI: 10.1080/01621459.1999.10473876