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Perturbations and zero points for equations with accretive mappings in fuzzy normed spaces

Perturbación y puntos cero para ecuaiones con mapeo acumulativo en espacios normados difusos

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Abstract

The purpose of this paper is to investigate the 1-set-contractive perturbations of accretive operators and discuss the solution of a special type of operator equations in fuzzy normed spaces. Also we shall study the perturbations, and the existence, problems of zero points for nonlinear equations with accretive mappings in fuzzy normed spaces.

Key words and phrases: accretive operator, iterative method, fixed point theorem, nonexpansive mapping, zero point.

Resumen

El propósito de este artículo es investigar las perturbaciones 1-conjunto contractivas de operadores acumulativos y discutir la solución de un tipo especial de ecuaciones de operadores en espacios normados difusos. También, estudiaremos las perturbaciones y existencia de problemas de puntos cero para ecuaciones no lineales con mapeo acumulativo en espacios normados difusos.

Palabras y frases clave: operador acumulativo, método iterativo, teorema del punto fijo, mapeo no expansivo, punto cero.

1 Introduction

It is well known that the concept of fuzzy metric space, which was initiated by O. Kramosil and J. Michalek in 1975, is an important generalization of metric space, and the fixed point theory in fuzzy metric spaces has been studied by many authors. The topological degree is a fundamental concept in algebraic topology and in analysis, and the number of its applications to nonlinear differential equations has increased at an impressive rate during the whole second half of the 20th century.

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Li et al. introduced and studied the topological degree of 1-set-contractive fields in Banach spaces ([**?**]). Recently, by using the topological degree method, Li and Xu obtained many new results for 1-set-contractive operators in Banach space ([**?**, **?**]). The topological degree for compact continuous operators in PN-spaces was first defined by Chang and Chen ([**?**]). Since then, the topological degrees of compact continuous operators, k-set-contractive operators, condensing operators and the A-proper degree in PN-spaces and the corresponding fixed point theorems have been studied extensively ([**?**, **?**, **?**, **?**, **?**, **?**, **?**]). Also, the accretive (m-accreitve) operators in PN spaces were introduced and studied ([**?**, **?**]). In [**?**],they established the topological degree of 1-set-contractive fields in PN-spaces, and obtained some new fixed point theorems.

The purpose of this paper is to further investigate the 1-set-contractive perturbations of accretive operators and discuss the solution of a special type of operator equations in fuzzy normed spaces.

For the sake of convenience, we first recall some definitions, notations as well as some lemmas which are useful in proving our main results in Section 2.

Definition 1.1. [**?**] A binary operation $T : [0,1] \times [0,1] \to [0,1]$ is said to be a continuous t-norm if $([0,1], T)$ is a topological monoid with unit 1 such that $T(a,b) \leq T(c,d)$ whenever $a \leq c, b \leq d$ for all $a,b,c,d \in [0,1]$.

Some typical examples of t-norm are the following:

$$\begin{aligned}
&T(a,b) = ab, \quad \text{(product)}\\
&T(a,b) = \min\{a,b\}, \quad \text{(minimum)}\\
&T(a,b) = \max\{a+b-1,0\}, \quad \text{(Lukasiewicz)}\\
&T(a,b) = \frac{ab}{a+b-ab}, \quad \text{(Hamacher)}
\end{aligned}$$

Definition 1.2. [**?**] Let X be a vector space over a field $\mathbb{K}$ (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$) and T be a continuous t-norm. A fuzzy set N in $X \times [0,\infty)$ is called a fuzzy norm if it satisfies the following conditions:

(FN1:) $N(x,0) = 0$, for all $x \in X$;

(FN2:) $N(x,t) = 0$ for all $t > 0$ if and only if $x = 0$;

(FN3:) $N(\lambda x, t) = N\left(x, \frac{t}{|\lambda|}\right)$ for all $x \in X$ and all scalar $\lambda \neq 0$;

(FN4:) $N(x+y, t+s) \geq T(N(x,t), N(y,s)$ for all $x, y \in X$ and all $t, s > 0$;

(FN5:) for all $x \in X$, $N(x,.)$ is left continuous and $\lim_{t\to\infty} N(x,t) = 1$.

The triple (X, N, T) will be called fuzzy normed linear space (briefly, FNLS).

Lemma 1.1. *[**?**] Let (X,N,T) be a FNLS. Then $N(x,.)$ is non-decreasing, for all $x \in X$.*

Theorem 1.1. *[**?**] Let (X,N,T) be a FNLS. For $x \in X$, $r \in (0,1)$, $t > 0$, we define the open ball*

$$B_x(r,t) := \{y \in X : N(x-y,t) > r\}.$$

Then

$$\tau_A := \{A \subset X : x \in A \Longleftrightarrow \exists t > 0, r \in (0,1) : B_x(r,t) \subset A$$

is a topology on X. Moreover, if the t-norm T satisfies $\sup_{t\in(0,1)} T(t,t) = 1$, then (X, τ_N) is Hausdorff.

Theorem 1.2. *[?] Let (X, N, T) be a FNLS. Then (X, τ_N) is a metrizable topological vector space.*

Definition 1.3. [?] Let (X, N, T) be a FNLS and $\{x_n\}$ be the sequence in X.

1. The sequence $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that

$$\lim_{t\to\infty} N(x_n - x, t) = 1, \text{ for all } t > 0.$$

   In this case x is called the limit of the sequence $\{x_n\}$ and we denote $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.

2. The sequence $\{x_n\}$ is called Cauchy sequence if

$$\lim_{n\to\infty} N(x_{n+p} - x_n, t) = 1$$

   for all $t > 0$ and all $p \in \mathbb{N}$.

3. (X, N, T) is said to be complete if every Cauchy sequence in X is convergent to a point in X. A complete FNLS will be called a fuzzy Banach space.

Definition 1.4. Let (X, N, T) be a a fuzzy normed space and D be a subset of X. A mapping $A : D \to X$ is said to be compact if $\overline{A(D)}$ is a compact subset of X.

Lemma 1.2. *Let (X, N, T) be a fuzzy normed space, T is a t-norm satisfying $T(t,t) \geq t$ for all $t \in [0,1]$, Ω be a nonempty subset of X, $S : \Omega \to X$ be a compact continuous mapping. Then for any neighborhood of θ, $u(\epsilon, \lambda)$, $\epsilon > 0$, $\lambda > 0$, there exists a finite dimension-valued compact mapping $S_{\epsilon,\lambda}$ such that*

$$Sx - S_{\epsilon,\lambda} \in u(\epsilon, \lambda), \qquad x \in \Omega.$$

Lemma 1.3. *Let (X, N, T) satisfy all the conditions of Lemma **??**. Let Ω be a nonempty open subset of X and $S : \overline{\Omega} \to X$ be a compact continuous mapping. Then $R = I - S$ is a closed mapping.*

Definition 1.5. Let (X, N, T) be a fuzzy normed space, T is a t-norm satisfying $T(t,t) \geq t$ for all $t \in [0,1]$. Let Ω be a nonempty open subset of X and $S : \overline{\Omega} \to X$ be a compact continuous mapping. Let $R = I - S$ and $p \in X \setminus R(\partial\Omega)$. By Lemma **??**, R is a closed mapping, $R(\partial\Omega)$ is a closed subset of X, and, consequently, there exists a neighborhood of θ, $u(\epsilon, \lambda)$, such that

$$(p + u(\epsilon, \lambda)) \cap R(\partial\Omega) = \emptyset.$$

By Lemma **??**, there exists a finite dimension subspace $X^{(n)}$ of X with $p \in X^{(n)}$ and a continuous compact mapping $S_n : \overline{\Omega} \to X^{(n)}$ such that $N(Sx - S_n x, \epsilon) > 1 - \lambda$ for all $x \in \overline{\Omega}$. Letting $\Omega_n = \Omega \cap X^{(n)}$ and $R_n = I - S_n$, we are going to prove $p \notin R_n(\partial\Omega)$.

In fact, if there exists some $x_0 \in \partial\Omega$ such that $p = R_n x_0$, then we have

$$N(Rx_0 - p, \epsilon) = N(Sx_0 - R_n x_0, \epsilon) = N(Sx_0 - S_n x_0, \epsilon) > 1 - \lambda.$$

This contradicts $(p+u(\epsilon,\lambda))\cap R(\partial\Omega)=\emptyset$. Beside, since $\overline{(I-(I-S_n))(\Omega_n)}$ is a compact set, the topological degree $deg_n(R_n,\Omega_n,p)$ in finite dimensional space $X^{(n)}$ is significant. We define the *Leray-Schauder topological degree* of R as follows:

$$Deg(R,\Omega,p)=\deg_n(R_n,\Omega_n,p). \tag{1.1}$$

2 Accretive mappings in fuzzy normed spaces

Definition 2.1. Let (X,N,T) is a fuzzy normed space and A be a nonempty subset of X. The function

$$D_A(t)=\sup_{s<t}\inf_{x,y\in A}N(x-y,s), \qquad t\in\mathbb{R}$$

is called the fuzzy diameter of A. If we have $\sup_{t>0}D_A(t)=1$, then A is called a fuzzy bounded subset; if $\sup_{t>0}D_A(t)=0$, then A is called a fuzzy unbounded subset.

Remark 2.1. In the sequel, we call the nonnegative number $\alpha_A(t)=\sup\{\epsilon\geq 0:$ there exist finite subsets A_i, $i=1,\cdots,n$ such that $A\subset\cup_{i=1}^n A_i$ and $D_{A_i}(t)\geq\epsilon\}$ the fuzzy noncompactness measure of A.

The fuzzy noncompactness measure has many important basic properties. The following will be useful in the sequel:

(i) $\alpha_A(t)=1$ for all $t>0$ if and only if A is a *relatively compact set*.

(ii) Assume that $S:Dom(S)\subset X\to X$ is a mapping and A is a fuzzy bounded set of $Dom(S)$. If there exists a $k\in(0,1)$ such that

$$\alpha_{SA}(t)\geq\alpha_A\left(\frac{t}{k}\right), \quad t\in\mathbb{R},$$

then S is called a k-set contraction mapping. If for any fuzzy bounded set $A\subset Dom(S)$ with $\alpha_A(t)\neq 1$, $\alpha_{SA}(t)>\alpha_A(t)$ for all $t>0$, then S is called a *condensing mapping*.

Definition 2.2. Let (X,N,T) is a fuzzy normed space. Then

1. A mapping $S:Dom(S)\subset X\to 2^X$ is said to be *accretive* if

$$N(x-y,t)\geq N(x-y+\lambda(u-v),t), \quad u\in Sx, v\in Sy, x,y\in Dom(S), \lambda>0.$$

2. The mapping S is said to be *maximal accretive* if

$$N(x-y_0,t)\geq N(x-y_0+\lambda(u-v_0),t), \qquad x\in Dom(S),\lambda>0,u\in Sx,$$

then for any $y_0\in Dom(S)$, we have $v_0\in Sy_0$.

3. The mapping S is said to be *m-accretive* if S is accretive and $I+S$ is surjective.

4. The mapping S is said to be *strongly accretive* if there exists a $k\in(0,1)$ such that

$$N((\lambda-k)(x-y),t)\geq N((\lambda-1)(x-y)+u-v,t) \tag{2.1}$$

for all $\lambda>k$, $x,y\in Dom(S)$, $u\in Sx$, $v\in Sy$.

5. The mapping S is said to be *dissipative* (resp., *maximal dissipative*) if $(-S)$ is accretive (resp., maximal accretive).

Proposition 2.1. *Let (X, N, T) be a complete fuzzy normed space with $T(t,t) \geq t$ for all $t \geq 0$. Then S is m-accretive if and only if for any $\lambda > 0$, $I + \lambda S$ is surjective.*

Proof. The sufficient condition is obvious.
Necessity: Let S be a m-accretive mapping. Then $I+S$ is surjective. Hence for any given $y_0 \in X$, the equation $y_0 \in (I + \lambda S)x$ has a solution x if and only if $x = (I+S)^{-1}(\lambda^{-1}y_0 + \lambda^{-1}(\lambda-1)x)$. Now we define a mapping R as follows:

$$R : x \to X, \quad Rx = (I+S)^{-1}[\lambda^{-1}y_0 + \lambda^{-1}(\lambda-1)x]. \tag{2.2}$$

It follows from (**??**) that for each $x \in X, \lambda^{-1}y_0 + \lambda^{-1}(\lambda-1)x - Rx \in Sx$. By the accretiveness of S, we have

$$\begin{aligned} N(Rx - Ry, t) &\geq N((x-y)\lambda^{-1}(\lambda-1), t) \\ &= N\left(x - y, \frac{\lambda}{|\lambda - 1|}t\right), \quad x, y \in X. \end{aligned}$$

When $\lambda > 1/2$, $R : X \to X$ is a contraction mapping. By [**?**], R has a unique fixed point in X, i.e., there exists an $w \in X$ such that

$$w = Rw = (I+S)^{-1}[\lambda^{-1}y_0 + \lambda^{-1}(\lambda-1)w].$$

This means that for any given $y_0 \in X$, the equation $y_0 \in (I+S)(x)$ has a solution w and so $ran(I + \lambda S) = X$ for all $\lambda > 1/2$.

Similarly, by the induction, we can prove that $ran(I + \lambda^n S) = X$ for all $n \geq 1, \lambda > 1/2$. From this we can especially obtain $ran(I + \lambda S) = X$ for all $\lambda > 0$. This achieves the proof. □

Now, we give some properties of accretive mappings and their resolvents in fuzzy normed spaces.

Let (X, N, T) be a fuzzy normed space and let A be an accretive mapping in X. We put $J_r = (I + rA)^{-1}$ and $A_r = \frac{1}{r}(I - J_r)$ for every $r > 0$. Then $Dom(J_r) = ran(I + rA)$, $ran(J_r) = Dom(A)$ and $Dom(A_r) = Dom(J_r)$ for every $r > 0$.

Firstly, we consider the properties of J_r:

Lemma 2.1. *Let (X, N, T) be a fuzzy normed space. Then J_r is single-valued and*

$$N(J_r x - J_r y, t) \geq N(x - y, t)$$

for every $x, y \in Dom(J_r)$, $r > 0$ and $t \in \mathbb{R}$.

Proof. Let $x, y \in Dom(J_r)$, $r > 0$ and $t \in \mathbb{R}$. Suppose that $y_1, y_2 \in J_r x$. Since A is accretive in X,

$$\begin{aligned} N(y_1 - y_2, t) &\geq N(y_1 - y_2 + r(\frac{1}{r}(x - y_1) - \frac{1}{r}(x - y_2)), t) \\ &= N(0, t) = 1. \end{aligned}$$

Hence we have $N(y_1 - y_2, t) = 1$ and so $y_1 = y_2$. There exist $[x_1, y_1], [x_2, y_2] \in A$ such that $x = x_1 + ry_1$ and $y = x_2 + ry_2$ and thus $J_r x = x_1$, $J_r y = x_2$. Since A is accretive in X,

$$N(J_r x - J_r y, t) = N(x_1 - x_2, t) \geq N(x_1 - x_2 + r(y_1 - y_2), t) = N(x - y, t).$$

This achieves the proof. □

Proposition 2.2. *Let (X, N, T) be a fuzzy normed space. Then*

1. *If $T(t,t) \geq t$ for every $t \in [0,1]$, then we have*

$$N(\frac{1}{n}(J_r^n x - x), t) \geq N(J_r x - x, t)$$

for all $x \in Dom(J_r^n)$, $r > 0$, $t \in \mathbb{R}$ and $n = 1, 2, \cdots$.

2. *$\frac{r}{p}x + \frac{p-r}{p}J_p x \in Dom(J_r)$ and $J_p x = J_r(\frac{r}{p}x + \frac{p-r}{p}J_p x)$ for all $x \in Dom(J_p)$, $p, r > 0$ and $t \in \mathbb{R}$.*

3. *$N(J_p x - J_r y, t) \geq N(\frac{r}{p+r}(x - J_r y) - \frac{p}{p+r}(y - J_p x), t)$ for all $x \in Dom(J_p)$, $y \in Dom(J_r)$, $p, r > 0$ and $t \in \mathbb{R}$.*

Proof. (1) Let $x \in Dom(J_r^n)$, $r > 0$, $t \in \mathbb{R}$ and $n = 1, 2, \cdots$. By the assumption and Lemma **??**, we have

$$\begin{aligned} N(\frac{1}{n}(J_r^n x - x), t) &= N(J_r^n x - x, nt) \\ &\geq T(N(J_r^n x - J_r^{n-1}x, t), N(J_r^{n-1}x - x, (n-1)t)) \\ &\geq T(N(J_r^n x - J_r^{n-1}x, t), T(N(J_r^{n-1}x - J_r^{n-2}x, t), \cdots \\ &T(N(J_r^2 x - J_r x, t), N(J_r x - x, t), \cdots))) \\ &\geq T(N(J_r x - x, t), T(N(J_r x - x, t), \cdots, \\ &T(N(J_r x - x, t), N(J_r x - x)))) \\ &\geq N(J_r x - x, t). \end{aligned}$$

(2) The proof follows similarity as in [**?**].
(3) Let $x \in Dom(J_p)$, $y \in Dom(J_r)$, $p, r > 0$ and $t \in \mathbb{R}$. Putting $q = \frac{p-r}{p+r}$, by (2),

$$\begin{aligned} &\frac{q}{p}x + \frac{p-q}{p}J_p x \in Dom(J_q), \\ &J_p x = J_q\left(\frac{q}{p}x + \frac{p-q}{p}J_p x\right) = J_q\left(\frac{r}{p+r}x + \frac{p}{p+r}J_p x\right), \\ &\frac{q}{r}y + \frac{r-q}{r}J_r y, \\ &J_r y = J_q\left(\frac{q}{r}y + \frac{r-q}{r}J_r y\right) = J_q\left(\frac{p}{p+r}y + \frac{r}{r+p}J_r y\right). \end{aligned}$$

By Lemma **??**, we have

$$\begin{aligned} N(J_p x - J_r y, t) &= N\left(J_q\left(\frac{r}{p+r}x + \frac{p}{p+r}J_p x\right) - J_q\left(\frac{p}{p+r}y + \frac{p}{p+r}J_p y\right), t\right) \\ &\geq N\left(\frac{r}{p+r}(x - J_r y) - \frac{p}{p+r}(y - J_p x), t\right). \end{aligned}$$

□

Next, we consider the properties of A_r:

Proposition 2.3. *Let (X, N, T) be a fuzzy normed space. Then*

1. *If $T(t,t) \geq t$ for every $t \in [0,1]$, then we have*

$$N(A_r x - A_r y, t) \geq N(\frac{2}{r}(x-y), t)$$

for every $x, y \in Dom(J_r)$, $r > 0$ and $t \in \mathbb{R}$.

2. *$A_r x \in AJ_r x$ for every $x \in Dom(J_r)$ and $r > 0$, and*

$$N(A_r x, t) \geq \sup_{y \in Ax} N(y, t)$$

for every $x \in Dom(A) \cap Dom(J_r)$ and $r > 0$.

Proof. (1) Let $x, y \in Dom(J_r)$, $r > 0$ and $t \in \mathbb{R}$. Then by Lemma **??**,

$$\begin{aligned} N(A_r x - A_r y, t) &= N\left(\frac{1}{r}(x-y) - \frac{1}{r}(J_r x - J_r y), t\right) \\ &\geq T\left(N\left(\frac{1}{r}(x-y), \frac{t}{2}\right), N\left(\frac{1}{r}(J_r x - J_r y), \frac{rt}{2}\right)\right) \\ &\geq T\left(N\left(\frac{1}{r}(x-y), t\right), N\left(\frac{1}{r}(x-y), t\right)\right) \\ &\geq N\left(\frac{1}{r}(x-y), t\right). \end{aligned}$$

(2) Let $x \in Dom(J_r)$ and $r > 0$. By the definition, $A_r x \in AJ_r x$. Let $x \in Dom(A) \cap Dom(J_r)$ and $r > 0$. Suppose $y \in Ax$. There exists $[x_1, y_1] \in A$ such that $x = x_1 + ry_1$ and so $J_r x = x_1$. By Lemma **??**, we have

$$\begin{aligned} N(A_r x, t) &= N(x - J_r x, rt) = N(J_r(x+ry) - J_r x, rt) \\ &\geq N(x + ry - x, rt) = N(y, t). \end{aligned}$$

Thus, it follows that $N(A_r x, t) \geq \sup_{y \in Ax} N(y, t)$. This achieves the proof. □

Definition 2.3. Let (X, N, T) be a fuzzy normed space and let $A, B : X \to 2^X$ be operators. B is said to be an extension of A if $Dom(A) \subset Dom(B)$ and $Ax \subset Bx$ for every $x \in Dom(A)$. We denote it by $A \subset B$.

Proposition 2.4. *Let (X, N, T) be a fuzzy normed space. If A is an m-accretive operator of X, then A is a maximal accretive operator of X.*

Proof. Let B be accretive in X with $A \subset B$. Let $r > 0$ and $t \in \mathbb{R}$. Let $[x, y] \in B$. Since A is m-accretive in X, $x + ry \in ran(I + rA)$. There exists $[x_1, y_1] \in A$ such that $x + ry = x_1 + ry_1$. Since B is accretive and $[x_1, y_1] \in B$,

$$N(x - x_1, t) \geq N(x - x_1 + r(y - y_1), t) = N(0, t) = 1.$$

Hence we have $x = x_1$ and thus $y = y_1$. Therefore, $[x, y] \in A$, that is, $B \subset A$ and thus $A = B$. Consequently, A is maximal accretive in X. This achieves the proof. □

Proposition 2.5. *Let (X, N, T) be a fuzzy normed space and $[x_0, y_0] \in X \times X$. Then A is maximal accretive in X if and only if*

$$N(x - x_0, t) \geq N(x - x_0 + r(y - y_0), t)$$

for every $[x, y] \in A$, $r > 0$ and $t \in \mathbb{R}$ implies $[x_0, y_0] \in A$.

Proof. Let A be maximal accretive in X. Put $\widehat{A} = A \cup [x_0, y_0]$. Then A is accretive in X and $A \subset \widehat{A}$. Since $\widehat{A}$ is maximal accretive in X, $\widehat{A} = A$. Hence $[x_0, y_0] \in A$. Conversely,let B be accretive in X with $A \subset B$. Let $[u, v] \in B$. Since B is accretive in X, for every $[x, y] \in A, r > 0$ and $t \in \mathbb{R}$, we have

$$N(x - u, t) \geq N(x - u + r(y - v), t).$$

By the assumption, $[u, v] \in A$ and so $B \subset A$. Hence $A = B$. Therefore A is maximal accretive in X. □

Proposition 2.6. *Let (X, N, T) be a fuzzy normed space and A be accretive in X. Then there exists a maximal accretive operator containing A.*

Proof. Let $\mathcal{B} = \{B : B \text{ is accretive in } X \text{ and } A \subset B\}$. Then $(\mathcal{B}, \subset)$ is a partially ordered set. Let $\mathfrak{T}$ be a totally ordered set with $\mathfrak{T} \subset \mathcal{B}$. it is easy to show that $\mathfrak{T}$ has an upper bound. By Zorn's lemma, there exists a maximal element in $\mathcal{B}$. This is a maximal accretive operator of X containing A. □

Next, consider the closeness of accretive operators

Proposition 2.7. *Let (X, N, T) be a fuzzy normed space and let A be accretive in X. Then the closure $\overline{A}$ of A is also accretive in X.*

Proof. Let $[x_1, y_1], [x_2, y_2] \in \overline{A}$. Then there exist $[x_{1n}, y_{1n}], [x_{2n}, y_{2n}] \in A$ such that $x_{1n} \to x_1, x_{2n} \to x_2, y_{1n} \to y_1$ and $y_{2n} \to y_2$. Let $r > 0$ and $t \in \mathbb{R}$. Since A is accretive,

$$N(x_{1n} - x_{2n}, t) \geq N(x_{1n} - x_{2n} + r(y_{1n} - y_{2n}), t).$$

Since N is lower semi-continuous on X, as $n \to \infty$, we have

$$N(x_1 - x_2, t) \geq N(x_1 - x_2 + r(y_1 - y_2), t).$$

Hence, $\overline{A}$ is accretive in X. This achieves the proof. □

Proposition 2.8. *Let (X, N, T) be a complete fuzzy normed space and let T be continuous with $T(t, t) \geq t$ for every $t \in [0, 1]$. Let A be accretive in X. If A is closed, then $ran(I + rA)$ is also closed fir every $r > 0$.*

Proof. Let $z_n \in ran(I + rA)$ such that $z_n \to z$. Then by assumption, $\{z_n\}$ is also a Cauchy sequence in X. There exists $[x_n, y_n] \in A$ such that $x_n + ry_n = z_n$ and so $J_r z_n = x_n$. Since A is accretive, for every $t \in \mathbb{R}$, we have

$$N(x_n - x_m, t) = N(J_r z_n - J_r z_m, t) \geq N(z_n - z_m, t).$$

Hence, it follows that

$$\underline{\lim}_{n,m\to\infty} N(x_n - x_m, t) \geq \underline{\lim}_{n,m\to\infty} N(z_n - z_m, t) = N(0, t) = 1$$

for every $t > 0$. Thus $\lim_{n,m\to\infty} N(x_n - x_m, t) = 1$ for every $t > 0$. Therefore, $\{x_n\}$ is a Cauchy sequence in X. there exists $x \in X$ such that $x_n \to x$ and so $y_n = \frac{1}{r}(z_n - x_n) \to \frac{1}{r}(z - x)$. Since A is closed, $[x, \frac{1}{r}(z-x)] \in A$. Hence $z \in x + rAx \in r(I + rA)$. Therefore, $ran(I + rA)$ is closed. This achieves the proof. □

Proposition 2.9. *Let (X, N, T) be a fuzzy normed space and A be maximal accretive in X. Then A is closed.*

Proof. Let $[x_n, y_n] \in A$ and $x_n \to x_0$, $y_n \to y_0$. Let $r > 0$ and $t \in \mathbb{R}$. Since A is accretive, for every $[x, y] \in A$, we have

$$N(x - x_n, t) \geq N(x - x_n + r(y_n - y_0), t).$$

Since N is lower semi-continuous on X, as $n \to \infty$,

$$N(x - x_0, t) \geq N(x - x_0 + r(y - y_0), t).$$

Since A is maximal accretive, by Proposition **??**, $[x_0, y_0] \in A$. Hence A is closed. This achieves the proof. □

Corollary 2.1. *Let (X, N, T) be a fuzzy normed space. Then*

1. *If A is m-accretive in X, then A is closed.*

2. *If A is maximal accretive in X, then Ax is a closed subset of X for every $x \in Dom(A)$.*

Proposition 2.10. *Let (X, N, T) be a fuzzy normed space and A be accretive in X. Let C be a closed convex subset of X and $p > r > 0$. If $C \subset ran(I+rA)$ and $J_r C \subset C$, then $C \subset ran(I+pA)$ and $J_p C \subset C$.*

Proof. Let $x \in C$ and $p > r > 0$. Define $S : C \to C$ by $Sz = J_r\left(\frac{r}{p}x + \frac{p-r}{p}z\right)$ for every $z \in C$. Let $t \in \mathbb{R}$. By Lemma **??**, for every $z_1, z_2 \in C$, we have

$$\begin{aligned}
N(Sz_1 - Sz_2, t) &= N\left(J_r\left(\frac{r}{p}x + \frac{p-r}{p}z_1\right) - J_r\left(\frac{r}{p}x + \frac{p-r}{p}z_2\right), t\right) \\
&\geq N\left(\frac{p-r}{p}(z_1 - z_2), t\right).
\end{aligned}$$

Since $0 < \frac{p-r}{p} < 1$, by [**?**], there exists a point $z \in C$ uniquely such that $Sz = z$. It follows that $x \in z + pAz \subset ran(I + pA)$. Thus $C \subset ran(I + pA)$ and $J_p C \subset C$. This achieves the proof. □

Finally, we consider the convergence of resolvents of accretive mappings in fuzzy normed spaces.

Proposition 2.11. *Let (X, N, T) be a fuzzy normed space and J_r be the resolvent of an accretive operator A for every $r > 0$ and T be continuous. Then*

$$\lim_{r\to 0^+} J_r x = x, \quad x \in \bigcap_{r>0} Dom(J_r) \cap Dom(A).$$

Proof. Let $x \in \bigcap_{r>0} Dom(J_r) \cap Dom(A)$ and $t \in \mathbb{R}$. By (2) of Proposition **??**, as $r \to 0^+$,

$$N(J_r x - x, t) = N\left(A_r x, \frac{t}{r}\right) \geq \sup_{y \in Ax} N\left(y, \frac{t}{r}\right) \to 1$$

for every $t > 0$. Thus we have $\lim_{r\to 0^+} N(J_r x - x, t) = 1$ for every $t > 0$. Hence $\lim_{r\to 0^+} J_r x = x$. □

Proposition 2.12. *Let (X, N, T) be a fuzzy normed space and J_r be the resolvent of an accretive operator A for every $r > 0$ and $T(t, t) \geq t$ for every $t \in [0, 1]$. Then*

$$\underline{\lim}_{r\to\infty} N\left(\frac{J_r x}{r}, t\right) = \lim_{r\to\infty} N(A_r x, t) = \sup_{y \in ran(A)} N(y, t)$$

for every $x \in \bigcap_{r>0} Dom(J_r)$ and $t \in \mathbb{R}$.

Proof. Let $x \in \bigcap_{r>0} Dom(J_r)$ and $t \in \mathbb{R}$. Put $d_t = \sup_{y\in ran(A)} N(y, t)$. Since $A_r x \in AJ_r x \subset ran(A)$ by (2) of Proposition **??**, $N(A_r x, t) \geq \sup_{y\in ran(A)} N(y, t) = d_t$. Thus $\overline{\lim}_{r\to\infty} N(A_r x, t) \geq d_t$. Let $\alpha \in (0, 1)$. By the definition of $d_{\alpha t}$, for every $\epsilon > 0$, $d_{\alpha t} - \epsilon < N(y_0, \alpha t)$ for some $[x_0, y_0] \in A$. By Proposition **??**, we have

$$\begin{aligned} N(A_r x, t) &= N(A_r x - A_r x_0 + A_r x_0, (1-\alpha)t + \alpha t) \\ &\geq T(N(A_r x - A_r x_0, (1-\alpha)t), N(A_r x_0, \alpha t)) \\ &\geq T\left(N\left(\frac{2}{r}(x - x_0), (1-\alpha)t\right), N(A_r x_0, \alpha t)\right). \end{aligned}$$

Thus, it follows that

$$\begin{aligned} \underline{\lim}_{r\to\infty} N(A_r x, t) &\geq \underline{\lim}_{r\to\infty} T\left(N\left(\frac{2}{r}(x - x_0), (1-\alpha)t\right), N(A_r x_0, \alpha t)\right) \\ &\geq T\left(\underline{\lim}_{r\to\infty} N\left(\frac{2}{r}(x - x_0), (1-\alpha)t\right), \underline{\lim}_{r\to\infty} N(A_r x_0, \alpha t)\right) \\ &= T(1, \underline{\lim}_{r\to\infty} N(A_r x_0, \alpha t)) \\ &\geq \sup_{y\in Ax_0} N(y, \alpha t) \geq N(y, \alpha t) > d_{\alpha t} - \epsilon. \end{aligned}$$

Since ϵ is arbitrary, as $\epsilon \to 0^+$, $\underline{\lim}_{r\to\infty} N(A_r x, t) \geq d_{\alpha t}$. Since $\lim_{\alpha\to 1^-} d_{\alpha t} = d_t$, $\underline{\lim}_{r\to\infty} N(A_r x, t) \geq d_t$. Therefore, $\lim_{r\to\infty} N(A_r x, t) = d_t$. The second equality holds.

Next, consider the first equality. Let $\alpha \in (0, 1)$. From

$$N\left(\frac{J_r x}{r}, t\right) = N\left(A_r x - \frac{x}{r}, t\right) \geq T\left(N(A_r x, \alpha t), N\left(\frac{x}{r}, (1-\alpha)t\right)\right),$$

we have

$$\begin{aligned} \underline{\lim}_{r\to\infty} N\left(\frac{J_r x}{r}, t\right) &\geq \underline{\lim}_{r\to\infty} T\left(N(A_r x, \alpha t), N\left(\frac{x}{r}, (1-\alpha)t\right)\right) \\ &\geq T\left(\underline{\lim}_{r\to\infty} N(A_r x, \alpha t), \underline{\lim}_{r\to\infty} N\left(\frac{x}{r}, (1-\alpha)t\right)\right) \\ &\geq \underline{\lim}_{r\to\infty} N(A_r x, \alpha t). \end{aligned}$$

As $\alpha \to 1^-$, $\underline{\lim}_{r\to\infty} N\left(\frac{J_r x}{r}, t\right) \geq \underline{\lim}_{r\to\infty} N(A_r x, t)$. Similarly, $\underline{\lim}_{r\to\infty} N(A_r x, t) \geq \underline{\lim}_{r\to\infty} N\left(\frac{J_r x}{r}, t\right)$. $\underline{\lim}_{r\to\infty} N(A_r x, t) = \underline{\lim}_{r\to\infty} N\left(\frac{J_r x}{r}, t\right)$. This achieves the proof. □

3 Perturbations and Zero Points for Equations with Accretive Mappings in Fuzzy normed spaces

In this section, we shall study the perturbation and the existence problems of zero points for nonlinear equations with accretive mappings in fuzzy normed spaces. In the sequel we always assume that (X, N, T) is a fuzzy normed space and T is a continuous t-norm satisfying $\sup_{0<t<1} T(t,t) = 1$.

Lemma 3.1. *Let D be a nonempty open set of X and $S : \overline{D} \to 2^X$ be a strongly accretive mapping.*

1. *Let $C = \{x \in D :$ there exists $t < 0$ such that $tx \in Sx\}$. If $\theta \in D$, then C is fuzzy bounded,*
2. *Let $u_n \in Sx_n$ and $\{x_n - u_n\}$ be fuzzy bounded. If $t_n \in (0,1]$ and $t_n \to t_0$, $z_n = (1-t_n)x_n + t_n$, $u_n \to y$, then $\{x_n\}$ is a Cauchy sequence of X.*

Proof. (1) If $x \in C$, then there exists a $t < 0$ such that $tx \in Sx$. Since S is strongly accretive, we have

$$N((\lambda - k)(x-\theta), s) \geq N((\lambda - 1)(x - \theta) + (tx - v), s) \tag{3.1}$$

for all $\lambda > k$, $k \in (0,1)$ and $v \in S\theta$ is a given point. Let $\lambda = 1 - t$ and so $\lambda > k$. By (**??**), we have

$$N((1-t-k)x, s) \geq N(-v, s) = N(v, s)$$

and so $N(v,s) \geq N(v, (1-t-k)s) \geq N(v, (1-k)s)$. This implies that C is fuzzy bounded.
(2) Since

$$N((\lambda - k)(x_n - x_m), s) \geq N((\lambda - 1)(x_n - x_m) + (u_n - u_m), s) \tag{3.2}$$

for all $\lambda > k$, letting $\lambda = t_n^{-1}$ and substituting it into (**??**), we have

$$N((1-kt_n)t_n^{-1}(x_n - x_m), s) \geq N((1-t_n)x_n + t_n u_n - (1-t_n)x_m - t_n u_m, t_n s).$$

This implies that

$$N(x_n - x_m, s) \geq N(z_n - z_m + (t_m - t_n)(x_m - u_m), (1-kt_n)s).$$

So, we have

$$N(x_n - x_m, s) \geq T\left(N\left(z_n - z_m, \frac{1-kt_n}{2}s\right), N\left(x_m - u_m, \frac{1-kt_n}{t_m - t_n}.\frac{s}{2}\right)\right).$$

Hence we have $\lim_{n,m\to\infty} N(x_n - x_m, s) = 1$ for all $s > 0$ and so $\{x_n\}$ is a Cauchy sequence of X. This achieves the proof. □

Theorem 3.1. *Let (X, N, T) be a complete fuzzy normed space with a continuous t-norm T, $D \subset X$ be an open subset and $S : \overline{D} \to X$ a single-valued continuous strongly accretive mapping. Suppose further that the following conditions are satisfied:*

(i) S maps a fuzzy bounded set into a fuzzy bounded set,

(ii) for any $t \in [0,1]$, $tI + (1-t)S$ is an open mapping,

(iii) there exists some $z \in D$ such that for each $x \in \partial D$ and each $t < 0$, $t(x-z) \neq Sx$.

Then the equation $Sx = \theta$ has a solution in $\overline{D}$.

Proof. Without loss of generality, we can assume that $z = \theta$ (otherwise, we can make a translation for D and S). Define a mapping $L_t : \overline{D} \to X$ by

$$L_t(x) = (1-t)x + tSx, \qquad t \in [0,1],$$

and let $M = \{t \in [0,1] : \text{there exists } x \in D \text{ such that } \theta = q_t(x)\}$. It is obvious that $\theta \in M$ and so $M \neq \emptyset$.

Now we prove that M is a closed set. In fact, let $\{t_n\}$ be a sequence in M and $t_n \to t_0$. Hence there exists $x_n \in D$ such that $\theta = L_{t_n}(x_n), n = 1, 2, \cdots$. Then we have

$$\frac{1-t_n}{t_n} x_n = Sx_n, \quad n = 1, 2, \cdots.$$

By Lemma **??**, $\{x_n\}$ is fuzzy bounded and so $\{x_n - Sx_n\}_{n=1}^{\infty}$ is a fuzzy bounded set. By Lemma **??**, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Let $x_n \to x_0$. Hence $(1-t_0)x_n + t_0 Sx_0 = \theta$, i.e., $t_0 \in M$. This shows that M is a closed set.

Now we use the method of reduction to absurdity to prove $\sup M = 1$. If $\sup M \neq 1$, then there exist $t_1 \in M$ and a sequence $\{t_n\} \subset [0,1]$, $t_n \in M$ such that $t_n \to t_1$. Since $\theta = L_{t_1}(x_1)$, where x_1 is a point in D, let C be an open neighborhood of x_1 and $C \subset D$. It is obvious that

$$y_n = L_{t_n} \in L_{t_n}(C), \quad n = 1, 2, \cdots.$$

Since $\theta \notin L_{t_n}(C)$ and $L_{t_n}(C)$ is an open set, $\{ty_n : t \in [0,1]\} \cap L_{t_n}(\partial C) \neq \emptyset$. Now we prove that $L_{t_n}(\overline{D})$ is a closed set for $n = 1, 2, \cdots$. In fact, if $(1-t_n)x_1 + t_n Sx_1 \to z$ as $n \to \infty$, it follows from

$$N((1-t_n k)t_n^{-1}(x_{i1} - x_{i2}), t) \geq N((1-t_n)t_n^{-2}(x_{i1} - x_{i2}) + (Sx_{i1} - Sx_{i2}), t)$$

for all $t > 0$ that

$$N((1-t_n k)(x_{i1} - x_{i2}), t) \geq N(L_{t_n}(x_{i1}) - L_{t_n}(x_{i2}), t)$$

for all $t > 0$. Hence we have $x_i \to x_0 \in \overline{C}$ and so $z = L_{t_n}(x_0) \in L_{t_n}(\overline{C})$. This implies that $L_{t_n}(\overline{C})$ is a closed set.

Next, we have $\partial L_{t_n}(C) \subset L_{t_n}(\partial C)$ and there exists a point $x_n \in \partial C$ such that $L_{t_n}(x_n) \in \{ty_n : t \in [0,1]\}$. Since $y_n \to \theta$, $L_{t_n}(x_n) \to \theta$. Let $z_n = L_{t_n}(x_n) = (1-t_n)x_n + t_n Sx_n$. Since we have

$$N((\lambda - k)(x_n - \theta), s) \geq N((\lambda - 1)(x_n - \theta) - (1-t_n)t_n^{-1}x_n + t_n^{-1}z_n - v, s)$$

for all $\lambda > k$, $s > 0$, where $v = S\theta$, taking $\lambda = t_n^{-1}$, we have $N((1-t_n k)t_n^{-1}x_n, s) \geq N(t_n^{-1}(z_n - t_n v), s)$ for all $s > 0$, i.e.,

$$N(x_n, s) \geq N(x_n - t_n v, s(1-k)).$$

This implies that $\{x_n\}$ is fuzzy bounded and so $\{x_n - Sx_n\}$ is fuzzy bounded. By Lemma **??**(2), $x_n \to x_2 \in \partial C$ and hence we have

$$(1-t_1)x_2 + t_1 Sx_2 = \theta.$$

Since $N((\lambda-k)(x_2-x_1),s) \geq N((\lambda-1)(x_2-x_1)+Sx_2-Sx_1,s)$, letting $\lambda = t_1^{-1}$, we have

$$N((1-t_1)t_1^{-1}(x_2-x_1),s) \geq 1,$$

which implies that $x_1 = x_2$, which is a contradiction. Hence $\sup M = 1$ and so there exist $t_n \to a$, $x_n \in D$ such that $(1-t_n)x_n + t_n S x_n = \theta$.

By the same way stated above, we can prove that $\{x_n\}$ is fuzzy bounded and so $x_n \to x_0 \in \overline{D}$, $Sx_0 = \theta$. This implies that the equation $Sx = \theta$ has a solution in $\overline{D}$. This achieves the proof. □

Corollary 3.1. *Let (X,N,T) be a complete fuzzy normed space, Ω be an open set of X, $\theta \in \Omega$, $S : \overline{\Omega} \to X$ be a continuous strongly pseudo-contraction mapping, $I - tS$ an open mapping $I - tS$ be a mapping from a fuzzy bounded set into a fuzzy bounded set. Suppose that for each $x \in \partial\Omega$, $Sx \neq \lambda x$ for all $\lambda \geq 1$. Then S has a fixed point in Ω.*

Proposition 3.1. *Let (X,N,T) be a complete fuzzy normed space with $T(t,t) \geq t$ for all $t \in [0,1]$. Let $S : D \to 2^X$ be a strongly accretive mapping and D be a nonempty closed set of X. If $D \subset (I+S)(D)$, then S has a zero point in D.*

Proof. Since S is a strongly accretive mapping,

$$N((\lambda-k)(x-y),t) \geq N((\lambda-1)(x-y)+u-v,t)$$

for all $u \in Sx$, $v \in Sy$, $x,y \in D$, $\lambda > k$, $k \in (0,1)$. Letting $\lambda = 2$, we have

$$N((2-k)[(I+S)^{-1}z - (I+S)^{-1}w],t) \geq N(z-w,t)$$

for all $z,w \in (I+S)D$. This implies that $(I+S)^{-1} : D \to D$ is a contraction mapping and so there exists a fixed point in D, i.e., there exists x_0 such that $x_0 = (I+S)^{-1}x_0$. Hence we have $\theta \in Sx_0$. This achieves the proof. □

Corollary 3.2. *Let (X,N,T) be a complete fuzzy normed space with $T(t,t) \geq t$ for all $t \in [0,1]$. Let $S : D \to 2^X$ be a strongly accretive mapping and D be a nonempty closed set of X. If $(I+S)D = D$, then $ran(S) = X$.*

Proof. For any given $p \in X$, let $S_0 = S - p$. By Proposition **??**, the equation $p \in Sx$ has a solution in D. This achieves the proof. □

Remark 3.1. Assume that (X,N,T) is a complete fuzzy normed space with $T(t,t) \geq t$ for all $t \in [0,1]$, Ω is a nonempty open subset of X, $S_1 : \overline{\Omega} \to 2^X$ is accretive mapping, $S_2 : \overline{\Omega} \to X$ is a continuous condensing mapping such that $S_2(\overline{\Omega})$ is fuzzy bounded and $S_2(\overline{\Omega}) \subset (I+S_1)(\overline{\Omega})$. Now we consider the existence problem of solutions for the following multi-valued equation:

$$\theta \in (I+S_1-S_2)(x). \tag{3.3}$$

Since S_1 is accretive, $(I+S_1)^{-1}$ is a nonexpansion mapping. Therefore the equation (**??**) is equivalent to the following equation:

$$x \in (I+S_1)^{-1}S_2 x. \tag{3.4}$$

It is easy to know that $(I+S_1)^{-1}S_2$ is a condensing mapping.

If $S:\overline{\Omega}\to X$ is a continuous k-set contraction mapping, $k\in(0,1)$, $S(\overline{\Omega})$ is a fuzzy bounded set, $\theta\notin(I-S)(\partial\Omega)$, and denote

$$D=\bigcap_{n=1}^{\infty}D_n;\quad D_1=\overline{co}(S(\overline{\Omega})),\quad D_n=\overline{co}(D_{n-1}\cap\overline{\Omega}),n\geq 2.$$

If there exists an n_0 such that $D_{n_0}=\emptyset$, then we define the topological degree as:

$$\deg(I-S,\Omega,\theta)=0.$$

If D is a nonempty set of X, it is easy to see that D is a compact convex subset of X. By J. Dugundji [?], there exists a retraction $r:X\to D$. Letting

$$S_r=S.r:\overline{\Omega}\to D,$$

then S_r is a compact mapping and $\theta\notin(I-S_r)(\partial\Omega)$. The topological degree $\deg(I-S,\Omega,\theta)$ has meaning [?], and we define

$$\deg(I-S_r,\Omega,\theta)=\deg(I-S,\Omega,\theta).\tag{3.5}$$

It is easy to prove that the topological degree defined by (??) is well-defined (see [?, ?] and [?]).

If $S:\overline{\Omega}\to X$ is a continuous condensing mapping, $S(\overline{\Omega})$ is a fuzzy bounded set and $\theta\notin(I-S)(\partial\Omega)$, then there exists a $t_0>0$ such that

$$\sup_{x\in\partial\Omega}N(x-Sx,t_0)=\rho<1.$$

Since $S(\overline{\Omega})$ is fuzzy bounded, when $k\in(0,1)$ and k is sufficiently near 1, then we have

$$\inf_{x\in\partial\Omega}N\left(Sx,\frac{t_0}{2(1-k)}\right)>\rho.$$

Let $S_k=kS$. Then S_k is a k-set contraction mapping, $0<k<1$. When k is sufficently near 1, it follows from the following inequality:

$$N(x-Sx,t_0)\geq\min\left\{N\left(x-Sx,\frac{t_0}{2}\right),N\left(Sx,\frac{t_0}{2(1-k)}\right)\right\}$$

for all $x\in\partial\Omega$ that

$$N(x-Sx,t_0)\geq N\left(x-S_kx,\frac{t_0}{2}\right),\quad x\in\partial\Omega.$$

Therefore, we have $\theta\notin(I-S_k)(\partial\Omega)$ and so the topological degree $\deg(I-S_k,\Omega,\theta)$ is well-defined. We define

$$\deg(I-S,\Omega,\theta)=\deg(I-S_k,\Omega,\theta).\tag{3.6}$$

Now we turn to discuss the existence problem of solutions of equation (??). If $\theta\notin(I+S_1-S_2)(x)$ for all $x\in\partial\Omega$, then $x\notin(I+S_1)^{-1}S_2x$ for all $x\in\partial\Omega$. Hence the topological degree $\deg(I-(I+S_1)^{-1}S_2,\Omega,\theta)$ is well-defined.

Theorem 3.2. *Let (X, N, T) be a complete fuzzy normed space with $T(t,t) \geq t$ for all $t \in [0,1]$, $\Omega \subset X$ be an open subset and $\theta \in \Omega$. Suppose that $S_1 : \overline{\Omega} \to 2^X$ is an accretive mapping, $S_2 : \overline{\Omega} \to X$ is a continuous condensing mapping and $S_2(\overline{\Omega})$ is a fuzzy bounded set and for any $t \in (0,1]$, $tS_2(\overline{\Omega}) \subset (I + tS_1)(\overline{\Omega})$. If for any $x \in \partial\Omega$ and any $\lambda \geq 1$, $\lambda x \notin (S_2 - S_1)(x)$, then the equation $\theta \in (I + S_1 - S_2)(x)$ has a solution in Ω.*

Proof. Since $\lambda x \notin (S_2 - S_1)(x)$ for any $x \in \partial\Omega$ and any $\lambda \geq 1$ and S_1 is accretive, we have

$$x \neq (I + tS_1)^{-1}.tS_2 x, \quad x \in \partial\Omega.$$

Since $(I + tS_1)^{-1}.tS_2 : [0,1] \times \overline{\Omega} \to X$ is continuous condensing, the topological degree $\deg(I - (I + S_1)^{-1}.tS_2, \Omega, \theta)$ is well-defined and it is independent of $t \in [0,1]$. Hence we have

$$\deg(I - (I + S_1)^{-1} S_2, \Omega, \theta) = \deg(I - \theta, \Omega, \theta) = 1.$$

This implies that the equation $\theta \in (x + S_1 x - S_2 x)$ has a solution in Ω. This achieves the proof. □

Corollary 3.3. *Let (X, N, T), Ω be the same as in Theorem **??**. Suppose that $S_1 : \overline{\Omega} \to 2^X$ is an m-accretive mapping, $S_2 : \overline{\Omega} \to X$ is a continuous condensing mapping and $S_2(\overline{\Omega})$ is fuzzy bounded. If for any $x \in \partial\Omega$ and for any $\lambda \geq 1$, $\lambda x \notin (S_2 - S_1)(x)$. Then the equation $\theta \in (I + S_1 - S_2)(x)$ has a solution in Ω.*

Corollary 3.4. *Let (X, N, T), Ω be the same as in Theorem **??**. Suppose that $S_1 : \overline{\Omega} \to 2^X$ is an accretive mapping, $S_2 : \overline{\Omega} \to X$ is a continuous condensing mapping and $S_2(\overline{\Omega})$ is fuzzy bounded and for any $t \in (0,1]$, $tS_2(\overline{\Omega}) \subset (I + tS_1)(\overline{\Omega})$. If $N(S_2 x - f, t) \geq N(x, t)$ for all $x \in \partial\Omega$, $f \in S_1 x$, $t > 0$, then $\theta \in (I + S_1 - S_2)(\overline{\Omega})$.*

Proof. Without loss of generality, we can assume that $\theta \notin (I + S_1 - S_2)(\partial\Omega)$. if for some $x_0 \in \partial\Omega$ and some $\lambda > 1$ such that $\lambda x_0 \in S_2 x_0 - S_1 x_0$, then we have

$$N(\lambda x_0, t) \geq N(x_0, t), \qquad t > 0,$$

and so we have $x_0 = \theta$, which is a contradiction. Hence for any $x \in \partial\Omega$ and $\lambda \geq 1$, $\lambda x \notin (I + S_1 - S_2)(x)$. The conclusion follows from Theorem **??** immediately. This achieves the proof. □

Lemma 3.2. *Let (X, N, T) be a complete fuzzy normed space with $T(t,t) \geq t$ for all $t \in [0,1]$. Let Ω be an open subset of X, $\theta \in \Omega$. Suppose that $S_2 : \overline{\Omega} \to X$ is a continuous mapping and $\overline{S_2(\overline{\Omega})}$ is a compact set. Suppose that $S_1 : X \to X$ is a continuous dissipative mapping, $S_1(\partial\Omega)$ is a fuzzy bounded set and that $S_2(\overline{\Omega}) \subset (I - S_1)(X)$. If*

$$N(S_2 x + S_1 \mu x, t) \leq N(x, t), \quad t > 0, x \in \partial\Omega, \mu \in [0,1],$$

and for any $x \in \partial\Omega$, $x \neq (S_1 + S_2)x$, then

$$\deg(I - (I - S_1)^{-1} S_2, \Omega, \theta) = 0.$$

Proof. Since S_1 is dissipative, $(-S_1)$ is accretive. Next, since for any $x \in \partial\Omega$, $x \neq (S_1 + S_2)(x)$, the topological degree $\deg(I - (I - S_1)^{-1} S_2, \Omega, \theta)$ is well-defined. Now we prove that

$$\theta \notin \bigcup_{\mu \in [0,1]} [\mu I - (I - S_1)^{-1} S_2](\partial\Omega). \tag{3.7}$$

Suppose that this is not the case. Then there exist $\mu_n \to \mu_0$, $x_n \in \partial\Omega$ such that

$$-y_n = \mu_n x_n - (I - S_1)^{-1} S_2 x_n \to \theta.$$

Since we have $N(x_n, t) \geq N(S_1 x_n + S_2 x_n, t)$ for all $t > 0$, $\{x_n\}$ is fuzzy bounded. In the sequel, we discuss two cases:

(a) If $\mu_0 = 0$. Since $S_2 x_n = (I - S_1)(\mu_n x_n + y_n)$ and $y_n \to \theta$, $\mu_n x_n + y_n \to \theta$, by the continuity of S_1 and S_2, we have $S_2 x_n + S_1 \mu_n x_n \to \theta$. Besides, since $N(x_n, t) \geq N(S_2 x_n + S_1 \mu_n x_n, t) \to 1$ for all $t > 0$, $x_n \to \theta \in \partial\Omega$, which is a contradiction.

(b) If $\mu_0 \neq 0$. Since $\{(I - S_1)^{-1} S_2 x_n\}$ has a convergent subsequence, without loss of generality, we can assume that $(I - S_1)^{-1} S_2 x_n \to y_0$ and so $x_n \to \mu_0^{-1} y_0$. Hence we have

$$(I - S_1)^{-1} S_2 x_n \to y_0 = (I - S_1)^{-1} S_2(\mu_0^{-1}) y_0.$$

This shows that $y_0 = \mu_0(y_0/\mu_0) = S_1 \mu_0 (y_0/\mu_0) + S_2(\mu_0)^{-1} y_0$. It is obvious that $\mu_0 \neq 1$. Again since we have

$$N(y_0/\mu_0, t) \geq N(S_2(\mu_0)^{-1} y_0 + S_1 \mu_0 \mu_0^{-1} y_0, t) = N(\mu_0 . y_0/\mu_0, t)$$

for all $t \geq 0$, $N(y_0/\mu_0, t) = 1$ for all $t > 0$ and so we have $y_0 = \theta \in \Omega$, which is a contradiction.

Summing up the above discussion, we know that (**??**) is true. Hence by [**?**], we have $\deg(I - (I - S_1)^{-1} S_2, \Omega, \theta) = 0$. This achieves the proof.

□

Theorem 3.3. *Let (X, N, T) be the same as in Lemma **??**. Let Ω_1, Ω_2 be two open sets of X, $\theta \in \Omega_1 \subset \Omega_2$ and $\Omega_1 \neq \Omega_2$. Let $S_1 : X \to X$ be a continuous dissipative mapping, $S_2 : \overline{\Omega_2} \to X$ be a continuous mapping, $\overline{S_2(\overline{\Omega_2})}$ be a compact set and $tS_2(\overline{\Omega_2}) \subset (I - tS_1)(X)$ for all $t \in (0, 1]$. If one of the following conditions is satisfied:*

(i) For any $x \in \partial\Omega$, $N(x, t) \leq N(S_1 x + S_2 x, t)$ for all $t > 0$; for any $x \in \partial\Omega$, $\mu \in [0, 1]$, $N(S_2 x + S_1 \mu x, t) \leq N(x, t)$ for all $t > 0$ and $S_1(\partial\Omega_1)$ is fuzzy bounded,

(ii) for any $x \in \partial\Omega_2$, $\mu \in [0, 1]$, $N(S_2 x + S_1 \mu x, t) \leq N(x, t)$ for all $t > 0$ and $S_1(\partial\Omega_2)$ is fuzzy bounded; for any $x \in \partial\Omega_1$, $N(x, t) \leq N(S_1 x + S_2 x, t)$ for all $t > 0$.

Then $S_1 + S_2$ has a fixed point in $\overline{\Omega_2} \setminus \Omega_1$.

Proof. It suffices to prove that the conclusion is true under under the condition (i). Without loss of generality, we can assume that $S_1 + S_2$ has no fixed point on $\partial\Omega_1$ and $\partial\Omega_2$ (otherwise, the conclusion has been proved). From $N(x, t) \leq N(S_1 + S_2 x, t)$ for all $x \in \partial\Omega_2$ and for all $t > 0$, it follows that for any $x \in \partial\Omega_1$, $\lambda \geq 1$, $\lambda x \neq (S_1 + S_2)(x)$. Hence we have

$$\deg(I - (I - S_1)^{-1} S_2, \Omega_2, \theta) = 1.$$

By Lemma **??**, it follows that $\deg(I - (I - S_1)^{-1} S_2, \Omega_1, \theta) = 0$ and so

$$\deg(I - (I - S_1)^{-1} S_2, \Omega_2 \setminus \Omega_1, \theta) = 1.$$

This implies that $S_1 + S_2$ has a fixed point in $\overline{\Omega_2} \setminus \Omega_1$. This achieves the proof. □

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