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Mimetic methods to Helmholtz equation: numerical dispersion

Métodos miméticos para la ecuación de Helmholtz: dispersión numérica

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Abstract

This is the first in a series of papers in where mimetic finite difference methods (MFDM) to acoustic scattering are applying. A simple one-dimensional problem has been chosen to illustrate an implementation of MFDM. This problem consists of an incident flat pressure wave that disperses from an infinite rigid wall. An absorbent boundary condition is also applied. A study of the order of convergence and numerical dispersion of these methods has been carried out.

Key words and phrases: numerical dispersion, mimetic methods, acoustic scattering, Helmholtz equation.

Resumen

Este es el primero en una serie de artículos en donde los métodos de diferencias finitas miméticos (MFDM) son aplicados a la dispersión acústica. Se seleccionó un problema unidimensional simple para ilustrar una implementación de MFDM. Este problema consiste en una onda de presión plana incidente que se dispersa desde una pared rígida infinita. También es aplicada una condición de frontera absorbente. Se ha llevado a cabo un estudio del orden de convergencia y de la dispersión numérica de estos métodos.

Palabras y frases clave: dispersión numérica, métodos miméticos, dispersión acústica, ecuación de Helmholtz.

1 Introduction

The mimetic discretization methods are finite difference discretizations in which there are a set of discrete operators which maintain some important properties of the equivalent continuous operators. About a couple of decades ago, the support operators method was developed by Samarskii et al. [33, 34]. These discrete operators are build using a primal operator such that the second one is calculated using a Green's theorem discrete equivalent. Castillo and Grone [14]

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develop a matrix method in order to build mimetic discretizations such that they have high order approximations at all points in the grid for the divergence operator as well as the gradient operator on one dimensional staggered uniform grids. In the work presented by Montilla, Cadenas and Castillo [30], using the ideas of [10] this methodology to nonuniform staggered grids is extended, producing low order mimetic operators for the divergence and the gradient. Among other works related to the generation of mimetic operators can be mentioned [16, 19, 22, 35, 36, 37]. The mimetic methods have been applied both in the resolution of ordinary differential equations (boundary values problems) and in partial differential equations. Among them can be mentioned the following works [5, 6, 17, 20, 21, 26, 27, 28, 31, 38].To have a more general idea about the mimetic methods and some other applications it can be observed [15, 29].

Among the works related to the numerical resolution of the Helmholtz equation by mimetic methods, we have [12, 18]. On the study of the numerical dispersion of the numerical solution of various methods applied to wave problems can be mentioned [1, 2, 3, 4, 23, 24].

In this paper, it is show how to use the difference equations obtained from the mimetic operators for solving the Helmholtz equation in 1D, subject to the boundary conditions of rigid wall and irradiation at infinity. Section 2 is divided into three subsections which present the basics for developing this work. Subsection 2.1 describes the model problem, which will be analyzed along this work while subsections 2.2 and 2.3 will describe the notation used to designate the staggered grids, grid functions and discrete operators. Section 3 presents the mimetic finite difference equations obtained from the discretization process of the Helmholtz equation. Section 4 presents the order of convergence for different values of the wavenumber k using the support operator mimetic methods 1-2-1 and 2-2-2 for both, uniform and nonuniform grids. Section 5 presents a numerical and analytic study of the numerical pollution due to the dispersion of the scattered wave. Finally, section 6 presents some concluding remarks about the results obtained in this work.

2 Preliminars

In this section, it will be presented the governing equations of the one-dimensional acoustic wave scattering problem as well as an introduction to the generation of nonuniform staggered grids and grid functions. Additionally, the mimetic operators of divergence and gradient for staggered grids are here presented.

2.1 Model Problem

The one-dimensional acoustic scattering problem is modeled by the Helmholtz equation given by $p'' + k^2 p = 0$ which is subject to the rigid wall boundary condition $p'(0) = Ik$ and the irradiation condition $p'(1) - Ikp(1) = 0$ at infinity, where $I = \sqrt{-1}$. This elementary problem has been used by the author on several occasions in order to study the behavior of different methods, see for example [7, 8, 9, 11]. Applying the following variable change

$$z = p' \tag{1}$$

produces a first order system of differential equations of the form

$$\begin{aligned} z - p' &= 0 \quad (2)\\ z' + k^2 p &= 0 \quad (3) \end{aligned}$$

with boundary conditions

$$\begin{aligned} z(0) &= Ik && (4) \\ z(1) - Ikp(1) &= 0 && (5) \end{aligned}$$

The equation (2) comes from the variable change (1) that would represent the gradient $z = \nabla p$, likewise z' would represent the divergence $\nabla \cdot z$ in the equation (3).

2.2 Staggered Grid and Grid Functions

A uniform staggered grid was used in this work. Such a grid consists of N cells and $N+1$ nodes on the considered onedimensional domain. Any cell is define by the set of points between two consecutive nodes. It is usual to identify one cell by any of its interior points. In this case, every cell will be identified by its middle points. Therefore there will be N nodes x_i for $i = 1, 2, \ldots, N$ and $N-1$ cells $x_{i+\frac{1}{2}}$ for $i = 1, 2, \ldots, N$ where $x_{i+\frac{1}{2}} = \frac{x_i + x_{i+1}}{2}$. Three types of grid functions will be used; these are: nodal functions, cell-valued functions and extended functions. Nodal functions are defined as $f : HC \to HN$, cell-valued functions as $g : HN \to HC$ and, finally, the extended functions are defined as $g : HN \to HC^*$, where HC is the space of cell-valued functions, HN is the space of nodal functions and the symbol $*$ designates an extension to the space of cell-valued functions including the boundary nodes. For more detail on these definitions you can see [15].

2.3 Discrete Operators

Castillo and Yasuda [13] use the mimetic operators of divergence and gradient for the 1-2-1 and 2-2-2 methods. This operators have the form

$$\mathbf{D} = \frac{1}{h} \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \cdots & -1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & -1 & 1 \end{pmatrix} \tag{6}$$

$$\mathbf{G}_{\text{1-2-1}} = \frac{1}{h} \begin{pmatrix} -2 & 2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -2 & 2 \end{pmatrix} \tag{7}$$

and

$$\mathbf{G}_{2\text{-}2\text{-}2} = \frac{1}{h}\begin{pmatrix} -\frac{8}{3} & 3 & -\frac{1}{3} & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & \frac{1}{3} & -3 & \frac{8}{3} \end{pmatrix} \tag{8}$$

Divergence mimetic operator (6) is used in the 1-2-1 method as well as the 2-2-2 method because this operator has order two on the interior nodes and on the boundary ones as well. Gradient operator (7) is only used in the 1-2-1 method because it has order two on the interior nodes and order one at the boundary. The other gradient operator (8) has order two on the interior nodes as well as on the boundary nodes. This is the main difference between both methods.

3 Mimetic Finite Difference Equations

Let's denote $h_i = x_i - x_{i-1}$ and the approximations

$$z'_{i+\frac{1}{2}} = \frac{z_{i+1} - z_i}{h_{i+1}} \tag{9}$$

$$p'_i = \frac{p_{i+\frac{1}{2}} - p_{i-\frac{1}{2}}}{\frac{h_i + h_{i+1}}{2}} \tag{10}$$

which are all known [32] for the divergence as well as the gradient. Making a nonuniform discretization, given by $x_i = (a + \frac{b-a}{N}(i-1))^2$ as shown in [37], and substituting the approximations (9) and (10), the following difference equations are obtained

$$z_{i+1} - z_i + h_{i+1}k^2 p_{i+\frac{1}{2}} = 0 \quad i = 1, \cdots N-1 \tag{11}$$

$$\frac{h_i + h_{i+1}}{2} z_i - p_{i+\frac{1}{2}} + p_{i-\frac{1}{2}} = 0 \quad i = 1, \cdots N-1 \tag{12}$$

In order to include the boundary condition (4), (11) will be used with $i = 0$, knowing that $z_0 = Ik$, yielding the equation

$$z_1 + h_1 k^2 p_{\frac{1}{2}} = Ik \tag{13}$$

Similarly, in order to use the boundary condition (5) involved, the approximation $z_N = \frac{p_N - p_{N-\frac{1}{2}}}{\frac{h_N}{2}}$ is going to be used such that, solving for p_N and substituting it in (5) and knowing that $p_N = p(1)$, the following equation is also obtained

$$\left(\frac{h_N}{2} - \frac{1}{Ik}\right) z_N + p_{N-\frac{1}{2}} = 0 \tag{14}$$

In this way an algebraic system of $2N$ linear equations and $2N$ unknowns is given by (11-14). In order to illustrate the nature of this linear system of equations let us considerate the case for

$N = 4$

$$\begin{bmatrix} h_1k^2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \frac{h_1+h_2}{2} & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & h_2k^2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{h_2+h_3}{2} & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & h_3k^2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{h_3+h_4}{2} & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & h_4k^2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{h_4}{2} - \frac{1}{Ik} \end{bmatrix} \begin{bmatrix} p_{\frac{1}{2}} \\ z_1 \\ p_{\frac{3}{2}} \\ z_2 \\ p_{\frac{5}{2}} \\ z_3 \\ p_{\frac{7}{2}} \\ z_4 \end{bmatrix} = \begin{bmatrix} Ik \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{15}$$

When one uses the mimetic finite difference method 2-2-2 scheme, the coefficient matrix of the algebraic linear system of equations (15) given above is modified, such that only the last row of the system changes. In order to obtain this last equation, all the necessary calculations are shown below, using a uniform mesh.

First consider the gradient approximation at the right boundary $x = 1$ given by

$$z_N = \frac{\frac{3}{4}p_N - \frac{3}{2}p_{N-\frac{1}{2}} + \frac{1}{6}p_{N-\frac{3}{2}}}{\frac{h}{2}}$$

Now solving p_N for this equation and substituting it in (5) yields

$$\left(1 - \frac{3Ihk}{8}\right) z_N + \frac{Ik}{8} p_{N-\frac{3}{2}} - \frac{9Ik}{8} p_{N-\frac{1}{2}} = 0$$

In order to have an equivalent equation where only variables $p_{N-\frac{1}{2}}$ and z_N are involved, it is necessary to eliminate the variable $p_{N-\frac{3}{2}}$, which can be done by an algebraic summation of this equation with the equations (11) and (12), thus yielding

$$p_{N-\frac{1}{2}} - \frac{1 - Ihk/2}{Ik(h^2k^2+1)} z_N = 0 \tag{16}$$

Using the latter equation in a matrix structure like (15) and changing its last row for this equation, it can be clearly observed that the resulting matrix for 2-2-2 scheme is also symmetric (considerate the case for $N = 4$).

$$\begin{bmatrix} h_1k^2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \frac{h_1+h_2}{2} & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & h_2k^2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{h_2+h_3}{2} & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & h_3k^2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{h_3+h_4}{2} & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & h_4k^2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{Ihk-2}{2Ik(h^2k^2+1)} \end{bmatrix} \begin{bmatrix} p_{\frac{1}{2}} \\ z_1 \\ p_{\frac{3}{2}} \\ z_2 \\ p_{\frac{5}{2}} \\ z_3 \\ p_{\frac{7}{2}} \\ z_4 \end{bmatrix} = \begin{bmatrix} Ik \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

4 Order of Convergence

Having programmed these methods with MATLAB, various numerical tests were carried out, throwing an order of convergence of two for both, the pressure primary variable p and the velocity secondary variable z. These results are illustrated in Figures 1 and 2 for the 1-2-1 and 2-2-2 schemes respectively, using a uniform discretization. Surprisingly, the order of convergence

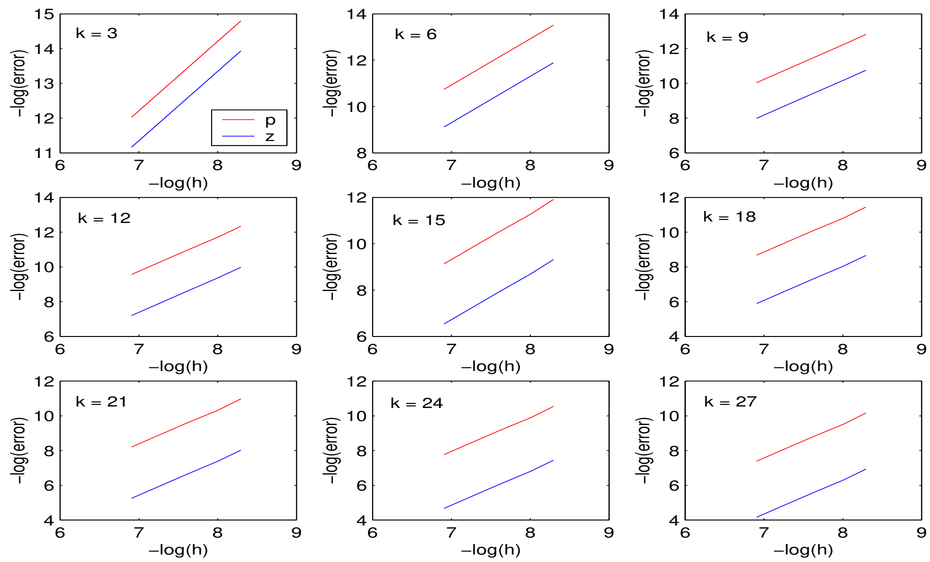

Figura 1: Order of convergence for the 1-2-1 scheme using a uniform discretization.

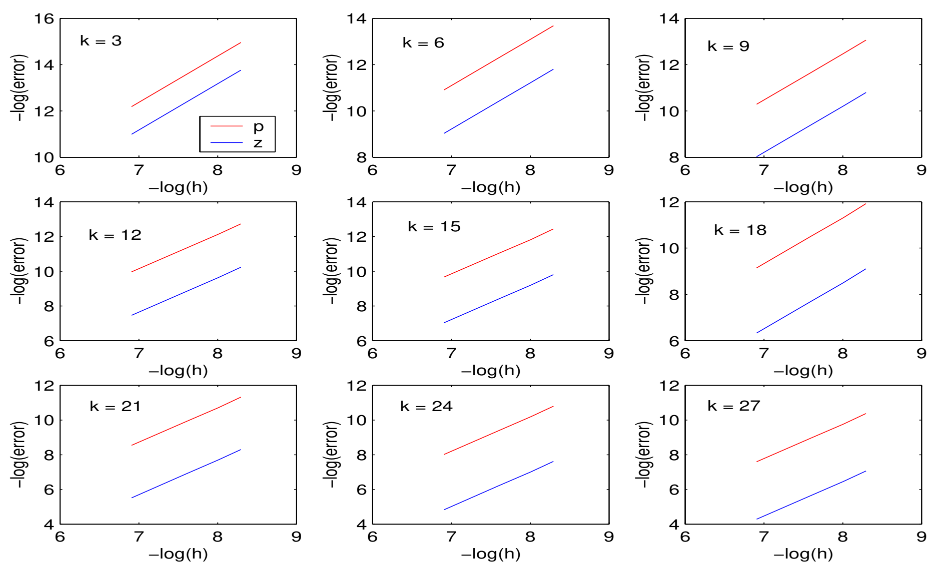

Figura 2: Order of convergence for the 2-2-2 scheme using a uniform discretization.

remains constant when a nonuniform discretization is used, differing from the results presented in [32], possibly because the differential equation herein considered is homogeneous.

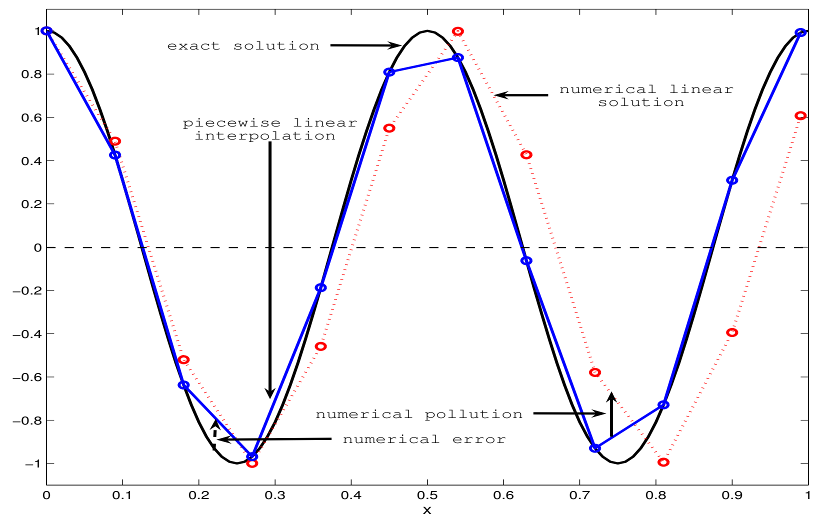

Figura 3: Effect of numerical dispersion in solving the Helmholtz equation.

5 Numerical Dispersion

An analysis of the difference between the analytical solution of the wavenumber and its numerical approximation will be done in this section. This difference is commonly refers to as the pollution error due to the dispersion. The total error is defined as the sum of the interpolating error plus the pollution error. This might be observed in Figure 3.

In order to calculate the wavenumber of the numerical solution for the numerical methods used in this research, a set of difference equations previously obtained (considering a uniform discretization) will be used. First the study will be done for the interior nodes, then for the nodes involved in the left border and later for the right border (both for the 1-2-1 method and for the 2-2-2 method).

5.1 Numerical Dispersion to the inner nodes

By using z_i from the equation (12) and replacing the result in (11) we get

$$p_{i+\frac{3}{2}} + (h^2k^2 - 2)p_{i+\frac{1}{2}} + p_{i-\frac{1}{2}} = 0 \tag{17}$$

which is solved using the basic procedures for solving an linear difference equation. For further details about these procedures, see [25].

In order to solve the ordinary linear difference equation with constants coefficients (17) of order two it is necessary to substitute

$$p_i = \lambda^i$$

such that one obtains a polynomial equation in λ of degree two, given by

$$\lambda^2 + (h^2k^2 - 2)\lambda + 1 = 0$$

Solving such an equation it is possible to obtain its two zeros:

$$\lambda_{1,2} = 1 - \frac{1}{2}(kh)^2 \pm \frac{kh}{2}\sqrt{(kh)^2 - 4} \tag{18}$$

Note that the values of λ are complex numbers for $0 < kh < 2$, being the cutoff frequency equal to two, which is not more than the value from which λ is real. It is also known that λ can be expressed as

$$\lambda = |\lambda| \left(\cos(\widetilde{k}h) \pm I \sin(\widetilde{k}h)\right) \tag{19}$$

where $\widetilde{k}$ is the wave number of the numerical solution. As can be seen from the equations (18) and (19) $\widetilde{k} \neq k$.

An estimate of the relative error, due to the effect of dispersion of the numerical solution, for the calculation of the numerical wavenumber is now presented. In order to do so, we consider the real part of the equations (18) and (19) and expanding $\arccos\left[\frac{Re(\lambda_1)}{|\lambda_1|}\right]$ into its Taylor series with $Re(\lambda_1) = 1 - \frac{1}{2}(kh)^2$ for $0 < kh \leq 2$ yields

$$\widetilde{k}h = kh + \frac{1}{24}(kh)^3 + \frac{3}{640}(kh)^5 + O((kh)^7) \tag{20}$$

from where it is obtained that the relative error in the calculation of k is given by

$$E_{r,k} = \frac{\widetilde{k}h - kh}{kh} = \frac{1}{24}(kh)^2 + \frac{3}{640}(kh)^4 + O((kh)^6) \tag{21}$$

Another fact when characterizing the wavenumber is the cutoff frequency, which is nothing else but the normalized frequency ($K = kh$), in which case the absolute value of λ abruptly changes, either expanding or contracting itself. In this case the cutoff frequency is two.

Figure 4 show the graphics of the real and imaginary part of λ as well as its absolute value. In both cases it can be seen that the absolute value of λ is one ($|\lambda_1| = |\lambda_2| = 1$) for $kh \leq 2$. If $kh > 2$ we have to $|\lambda_1|$ and $|\lambda_2| > 1$. We also have to $Re(\lambda_1) \approx \cos(kh)$ and $Im(\lambda_1) \approx \sin(kh)$ when $kh \leq 1$. The same happens for λ_2.

Figures 5 and 6 show a set of curves which relate $\tilde{k}$ to k for different values of h, and the percentage error for the approximation of k respectively. A single legend is used for all the graphics involved in these Figures.

5.2 Numerical Dispersion for the left boundary

In a similar way to how the calculations were made for the inner nodes, we proceed with the left boundary, for both the method 1-2-1 and for the method 2-2-2, because in both cases the equations involved are the same. Using the equations (12) and (13) with $i = 1$, we have

$$p_{\frac{3}{2}} + \left((kh)^2 - 1\right) p_{\frac{1}{2}} = Ikh$$

from where we get

$$\lambda^{\frac{3}{2}} + \left((kh)^2 - 1\right) \lambda^{\frac{1}{2}} = Ikh$$

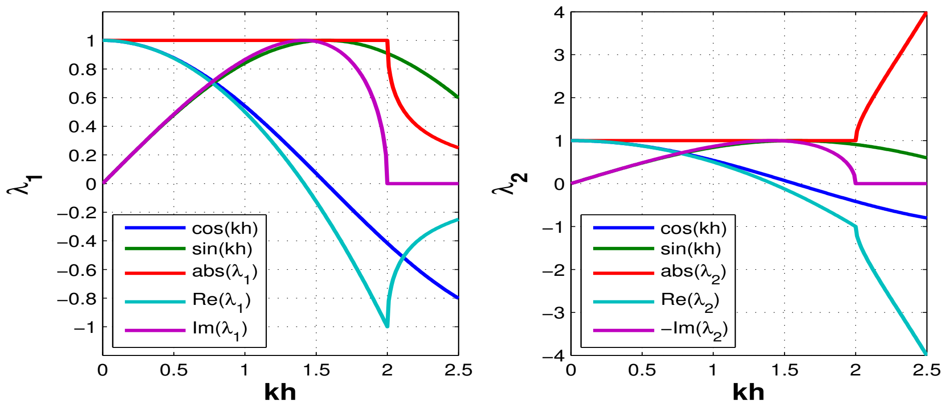

Figura 4: Real part, imaginary part and absolute value of λ.

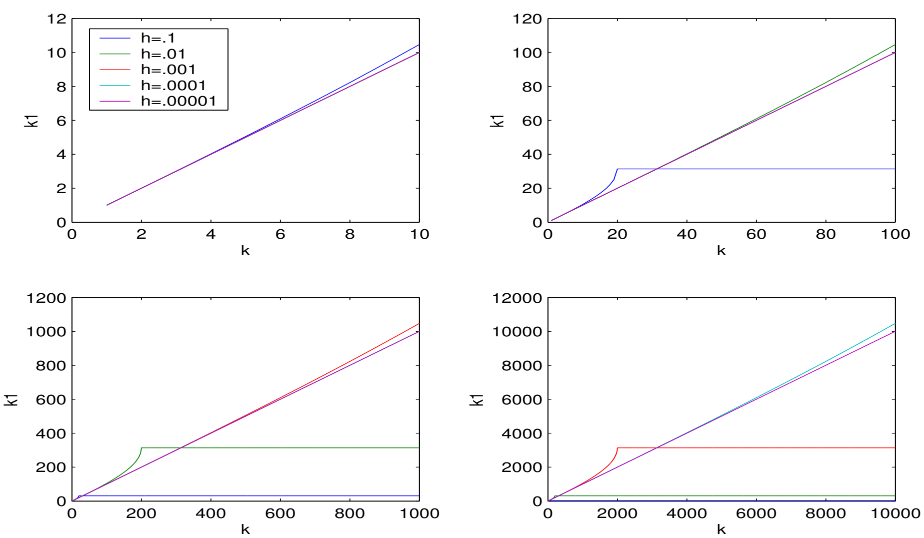

Figura 5: $\tilde{k}$ vs k for different values of h.

Whose roots are $\lambda_1^{\frac{1}{2}} = -Ikh$ and $\lambda_{2,3}^{\frac{1}{2}} = \frac{I}{2}\left(kh \pm \sqrt{(kh)^2 - 4}\right)$, thus

$$\lambda_1 = -(kh)^2 \quad \text{and} \quad \lambda_{2,3} = -\frac{1}{4}\left(kh \pm \sqrt{(kh)^2 - 4}\right)^2$$

λ_1 does not make physical sense and when simplifying $\lambda_{2,3}$, the same values are obtained as in the equation (18) and therefore the values of $\widetilde{kh}$ and the relative error in the calculation of k are given by (20) and (21).

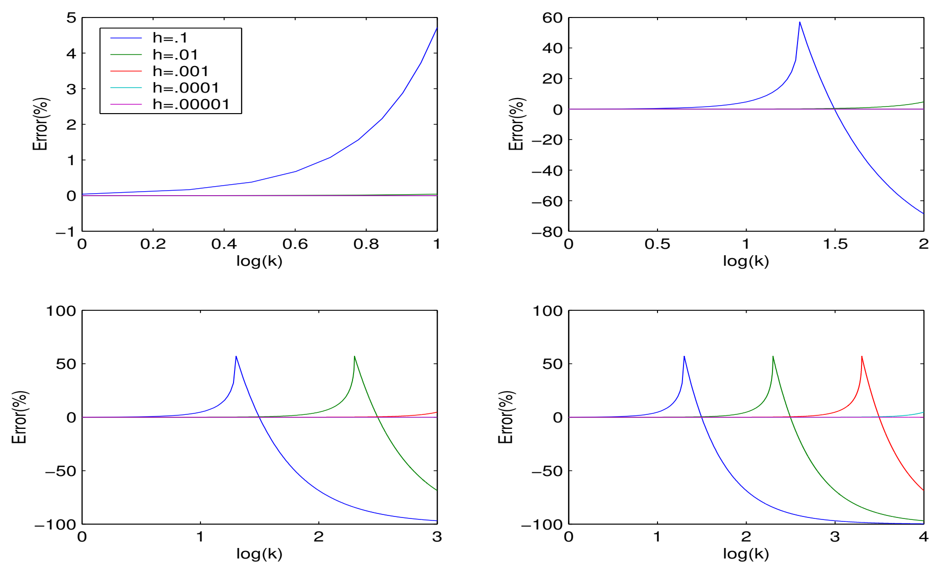

Figura 6: Relative error to the approximation of k.

5.3 Numerical Dispersion for the right boundary

In a similar way to how the calculations were made for the interior nodes, we proceed to the right boundary, both for method 1-2-1 and for method 2-2-2. Obtaining the following results

5.3.1 Method 1-2-1

From the equation (14) we have to

$$z_N = \frac{2Ik}{2 - Ikh} p_{N-\frac{1}{2}}$$

that when used in combination with the equations (11) and (12) with $i = N - 1$ we get

$$p_{N-\frac{3}{2}} + \left((kh)^2 - \frac{3kh + 2I}{kh + 2I} \right) p_{N-\frac{1}{2}} = 0$$

from where we get

$$\lambda = \frac{kh + 2I}{(kh + I)\,(2 - Ikh - (kh)^2)} = \frac{4 - (kh)^2 - (kh)^4 + I4kh}{(kh)^6 - 2(kh)^4 + (kh)^2 + 4} \tag{22}$$

Again considering the real part of the equations (22) and (19) and expanding $\arccos\left[\frac{Re(\lambda)}{|\lambda|}\right]$ into its Taylor series with $Re(\lambda) = \frac{4-(kh)^2-(kh)^4}{(kh)^6-2(kh)^4+(kh)^2+4}$ yields

$$\widetilde{k}h = kh - \frac{1}{3}(kh)^3 + \frac{77}{160}(kh)^5 + O((kh)^7)$$

from where it is obtained that the relative error in the calculation of k is given by

$$E_{r,k} = -\frac{1}{3}(kh)^2 + \frac{77}{160}(kh)^4 + O((kh)^6)$$

Graph on the left of Figure 7 show the related curves for the values of λ. It is emphasized that the Cutoff phenomenon does not appear. However $|\lambda| \approx 1$ only if $kh \ll 1$. Initially $|\lambda|$ grows and then decreases, observing areas where $|\lambda| > 1$ and where $|\lambda| < 1$. It also has $Re(\lambda) \approx \ cos(kh)$ and $Im(\lambda) \approx \sin(kh)$ if $kh < 1$.

5.3.2 Method 2-2-2

From the equation (16) we have to

$$z_N = \frac{2Ik}{2 - Ikh} p_{N-\frac{1}{2}}$$

that when used in combination with the equations (11) and (12) with $i = N - 1$ we get

$$p_{N-\frac{3}{2}} + \left((kh)^2 - 1 + \frac{I2kh\left((kh)^2 + 1\right)}{2 - Ikh} \right) p_{N-\frac{1}{2}} = 0$$

from where we get

$$\lambda = \frac{kh + 2I}{(kh)^3 - 2i(kh)^2 + 3kh + 2I} = \frac{(kh)^4 - (kh)^2 + 4 + I4kh\left((kh)^2 + 1\right)}{(kh)^6 + 10(kh)^4 + (kh)^2 + 4} \tag{23}$$

Again considering the real part of the equations (23) and (19) and expanding $\arccos\left[\frac{Re(\lambda)}{|\lambda|}\right]$ into its Taylor series with $Re(\lambda) = \frac{(kh)^4-(kh)^2+4}{(kh)^6+10(kh)^4+(kh)^2+4}$ yields

$$\widetilde{kh} = kh + \frac{13}{6}(kh)^3 - \frac{563}{163}(kh)^5 + O((kh)^7)$$

from where it is obtained that the relative error in the calculation of k is given by

$$E_{r,k} = \frac{13}{6}(kh)^2 - \frac{563}{163}(kh)^4 + O((kh)^6)$$

In the graph on the right of Figure 7 the related curves are shown for the values of λ. It is emphasized that the Cutoff phenomenon does not appear. It has $|\lambda| \approx 1$ only if $kh \ll 1$. $|\lambda|$ decreases monotonously, so $|\lambda| < 1$ for $kh > 0$. It also has $Re(\lambda_1) \approx \cos(kh)$ and $Im(\lambda_1) \approx \sin(kh)$ only if $kh \ll 1$.

6 Concluding Remarks

- A significative difference between the orders of convergence for the 1-2-1 and 2-2-2 mimetic finite difference methods, using uniform grids, is not observed.
- Both methods presented and kept order of convergence equals to two for the cases herein studied.

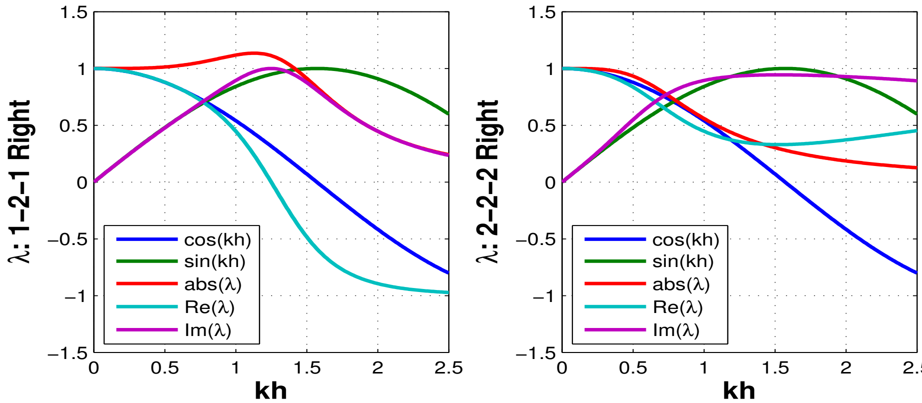

Figura 7: Real part, imaginary part and absolute value of λ.

- The cutoff frequency has a value of two for both the internal points and the left boundary for 1-2-1 and 2-2-2 methods.
- Cutoff frequency was not observed at the right boundary for none of the two methods studied.
- Relative error in the calculation of k both in the inner nodes and in the boundaries is $O((kh)^2)$.

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