Divulgaciones Matemáticas Vol. 19, No. 2 (2018), pp. 16–22

Coincidences in the Padovan and Tribonacci sequences

Coincidencias en las sucesiones de Padovan y Tribonacci

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Abstract

Let $(P_n)_{n\geqslant 0}$ be the *Padovan sequence* given by $P_0 = 0$, $P_1 = P_2 = 1$ and the recurrence formula $P_{n+3} = P_{n+1} + P_n$ for all $n \geqslant 0$. Let $(T_n)_{n\geqslant 0}$ be the *Tribonacci sequence* given by $T_0 = 0$, $T_1 = T_2 = 1$ and the recurrence formula $T_{n+3} = T_{n+2} + T_{n+1} + T_n$ for all $n \geqslant 0$. In this note we solve the Diophantine equation

$$P_n = T_m$$

in non-negative integers n, m. In particular, we find all the elements in the intersection of the Padovan and Tribonacci sequences.

Key words and phrases: Padovan, Tribonacci sequences, Linear forms in logarithms, reduction method.

Resumen

Sea $(P_n)_{n\geqslant 0}$ la *sucesión de Padovan* definida mediante $P_0 = 0$, $P_1 = P_2 = 1$ y la fórmula de recurrencia $P_{n+3} = P_{n+1} + P_n$ para todo $n \geqslant 0$. Sea $(T_n)_{n\geqslant 0}$ la *sucesión de Tribonacci* definida mediante $T_0 = 0$, $T_1 = T_2 = 1$ y la fórmula de recurrencia $T_{n+3} = T_{n+2} + T_{n+1} + T_n$ para todo $n \geqslant 0$. En este escrito resolvemos la ecuación Diofántica

$$P_n = T_m$$

en enteros no negativos n, m. En particular, encontramos todos los elementos en la intersección de las sucesiones de Padovan y Tribonacci.

Palabras y frases clave: Sucesiones de Padovan y Tribonacci, Formas lineales en logaritmos, Método de reducción.

Received 07/06/2018. Revised 14/08/2018. Accepted 20/11/2018.
MSC (2010): Primary 11J86; Secondary 11D61.
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1 Introduction

Let $(P_n)_{n\geqslant 0}$ be the *Padovan sequence*, named after the architect R. Padovan, given by $P_0 = 0$, $P_1 = P_2 = 1$ and the recurrence formula

$$P_{n+3} = P_{n+1} + P_n, \quad \text{for all} \quad n \geqslant 0. \tag{1.1}$$

This is sequence A000931 in [12]. The first few terms of this sequence are

$$0,\ 1,\ 1,\ 1,\ 2,\ 2,\ 3,\ 4,\ 5,\ 7,\ 9,\ 12,\ 16,\ 21,\ 28,\ 37,\ 49,\ 65,\ \ldots$$

Recently, some arithmetic properties of the Padovan sequence have been studied. Indeed, let $(F_n)_{n\geqslant 0}$ be the *Fibonacci sequence* given by the initial conditions $F_0 = 0$, $F_1 = 1$ and the recurrence formula $F_{n+2} = F_{n+1} + F_n$ for all $n \geqslant 0$. In [13], Stewart asks for the intersection of the Fibonacci and the Padovan sequences. In [14] De Weger solves this problem. He actually proves that the distance between Fibonacci and Padovan numbers growths exponentially. In [7] it is solved the problem of the intersection of the Padovan sequence and the powers of 2 and also the powers of 2 which can be written as sum of two terms of the Padovan sequence.

Now, let $(T_n)_{n\geqslant 0}$ be the *Tribonacci sequence* defined by $T_0 = 0$, $T_1 = T_2 = 1$ and the recurrence formula

$$T_{n+3} = T_{n+2} + T_{n+1} + T_n \tag{1.2}$$

holds for all $n \geqslant 0$. The first few terms of this sequence are

$$0,\ 1,\ 1,\ 2,\ 4,\ 7,\ 13,\ 24,\ 44,\ 81,\ 149,\ 274,\ 504,\ 927,\ \ldots$$

In this note, as a natural generalization of Stewart problem, we shall study the problem of the intersection of the Padovan and the Tribonacci sequences. More precisely, we will study the Diophantine equation

$$P_n = T_m \tag{1.3}$$

in non-negative integers n, m. Since $P_1 = P_2 = P_3 = 1$, $P_4 = P_5 = 2$ we assume that $n \neq 1, 2, 4$. That is, whenever we think of 1 and 2 as members of the Padovan sequence, we think of them as being P_3 and P_5, respectively. In the same way, as $T_1 = T_2 = 1$ we assume that $m \neq 1$. With these conventions, we prove the following result:

Theorem 1.1. *All non-negative solutions (n, m) of equation (1.3) belong to the set*

$$\{(0, 0), (3, 2), (5, 3), (7, 4), (9, 5)\}.$$

The intersection of the Tribonacci sequence with the Fibonacci one is studied in [9] and, it is a particular case in [3]. There are general results concerning the intersection of two linear recurrence sequences, see for example [11] and [2].

2 Tools

In this section, we gather the tools we need to prove Theorem 1.1. Let α be an algebraic number of degree d. Let a be the leading coefficient of its minimal polynomial over $\mathbb{Z}$ and let $\alpha^{(1)} = \alpha, \ldots, \alpha^{(d)}$ denote the conjugates of α. The *logarithmic height* of α is defined as

$$h(\alpha) = \frac{1}{d}\left(\log a + \sum_{i=1}^{d} \log \max\{\left|\alpha^{(i)}\right|, 1\}\right).$$

This height has the following basic properties. For α, β algebraic numbers and $m \in \mathbb{Z}$, we have:

- $h(\alpha+\beta) \leqslant h(\alpha) + h(\beta) + \log 2$.
- $h(\alpha\beta) \leqslant h(\alpha) + h(\beta)$.
- $h(\alpha^m) = |m| h(\alpha)$.

Now let $\mathbb{L}$ be a real number field of degree $d_{\mathbb{L}}$, $\alpha_1, \ldots, \alpha_\ell \in \mathbb{L}$ with $\alpha_i > 0$, $i = 1, \ldots, \ell$, and $b_1, \ldots, b_\ell \in \mathbb{Z}\backslash\{0\}$. Let $B \geqslant \max\{|b_1|, \ldots, |b_\ell|\}$ and

$$\Lambda = \alpha_1^{b_1} \cdots \alpha_\ell^{b_\ell} - 1.$$

Let $A_1, \ldots, A_\ell$ be real numbers such that

$$A_i \geqslant \max\{d_{\mathbb{L}} h(\alpha_i), |\log \alpha_i|, 0.16\} \qquad \text{for all} \qquad i = 1, \ldots, \ell.$$

The first tool we need is the following result due to Matveev in [10] (see also Theorem 9.4 in [5]).

Theorem 2.1. *Assume that $\Lambda \neq 0$. Then*

$$\log|\Lambda| > -1.4 \times 30^{\ell+3} \times \ell^{4.5} \times d_{\mathbb{L}}^2 (1 + \log d_{\mathbb{L}})(1 + \log B) A_1 \cdots A_\ell.$$

The second one, is a version of a reduction method of Baker-Davenport based on Lemma in [1]. We shall use the one given by Bravo, Gómez and Luca in [4]. For a real number x, we write

$$\|x\| = \min\{|x - n| \ : \ n \in \mathbb{Z}\}.$$

Lemma 2.1. *Let M be a positive integer. Let τ, μ, $A > 0$, $B > 1$ be given real numbers. Assume that the convergent p/q of τ is such that $q > 6M$ and $\varepsilon := \|q\mu\| - M\|q\tau\| > 0$. Then the inequality*

$$0 < |n\tau - m + \mu| < \frac{A}{B^w}$$

does not has a solution in positive integers n, m and w in the ranges

$$n \leqslant M \qquad \text{and} \qquad w \geqslant \frac{\log(Aq/\varepsilon)}{\log B}.$$

This lemma is a slightly variation of the one given by Dujella and Pethő in [6]. Finally, the following lemma is also useful. It is Lemma 6 in [8].

Lemma 2.2. *If $T > 3$ and $T > x/(\log x)$, then*

$$x < 2T \log T.$$

3 Proof of Theorem 1.1

We start with some properties of our sequences. For a complex number z we write $\overline{z}$ for its complex conjugate. Let $\omega \neq 1$ be a cubic root of 1. Put

$$\gamma := \sqrt[3]{\frac{9+\sqrt{69}}{18}} + \sqrt[3]{\frac{9-\sqrt{69}}{18}}, \quad \delta := \omega\sqrt[3]{\frac{9+\sqrt{69}}{18}} + \overline{\omega}\sqrt[3]{\frac{9-\sqrt{69}}{18}},$$

and

$$\alpha := \frac{1}{3}\left(1 + \sqrt[3]{19+3\sqrt{33}} + \sqrt[3]{19-3\sqrt{33}}\right),$$
$$\beta := \frac{1}{3}\left(1 + \omega\sqrt[3]{19+3\sqrt{33}} + \overline{\omega}\sqrt[3]{19-3\sqrt{33}}\right).$$

It is clear that $\gamma, \delta, \overline{\delta}$ are the roots of $X^3 - X - 1$ and, that $\alpha, \beta, \overline{\beta}$ are the roots of $X^3 - X^2 - X - 1$. These polynomials are both irreducible over $\mathbb{Q}$. It can be proved, by induction for example, the Binet formulas

$$P_n = c_1\gamma^n + c_2\delta^n + c_3\overline{\delta}^n, \qquad \text{and} \qquad T_n = d_1\alpha^n + d_2\beta^n + d_3\overline{\beta}^n \tag{3.1}$$

which hold for all $n \geqslant 0$, where

$$c_1 = \frac{\gamma(\gamma+1)}{2\gamma+3}, \qquad c_2 = \frac{\delta(\delta+1)}{2\delta+3}, \qquad c_3 = \overline{c_2} \tag{3.2}$$

and

$$d_1 = \frac{\alpha^2}{\alpha^2+2\alpha+3}, \qquad d_2 = \frac{\beta^2}{\beta^2+2\beta+3}, \qquad d_3 = \overline{d_2}. \tag{3.3}$$

The formulas (3.1) follows from the general theorem on linear recurrence sequences since the above polynomials are the characteristic polynomials of the Padovan and the Tribonacci sequences, respectively. We note that

$$\gamma = 1.32471\ldots, \; |\delta| = 0.86883\ldots, \; c_1 = 0.54511\ldots, \; |c_2| = 0.28241\ldots$$

and

$$\alpha = 1.83928\ldots, \; |\beta| = 0.73735\ldots, \; d_1 = 0.33622\ldots, \; |d_2| = 0.25999\ldots.$$

Further, the inequalities

$$\gamma^{n-3} \leqslant P_n \leqslant \gamma^{n-1}, \qquad \alpha^{n-2} \leqslant T_n \leqslant \alpha^{n-1} \tag{3.4}$$

also hold for all $n \geqslant 1$. These can be proved by induction.

Now we begin with the study of equation (1.3). If $n = 0$ then $0 = T_m$ implies $m = 0$. Thus we assumme that $n \geqslant 1$ and $m \geqslant 1$. By combining inequalities (3.4) we have

$$\gamma^{n-3} \leqslant P_n = T_m \leqslant \alpha^{m-1} \qquad \text{and} \qquad \gamma^{n-1} \geqslant P_n = T_m \geqslant \alpha^{m-2}. \tag{3.5}$$

From these we obtain

$$(n-3)\frac{\log\gamma}{\log\alpha} \leqslant (m-1) \qquad \text{and} \qquad (n-1)\frac{\log\gamma}{\log\alpha} \geqslant (m-2). \tag{3.6}$$

Since $\log\gamma/\log\alpha = 0.461453\ldots$ we have that if $n \leqslant 200$ then $m \leqslant 93$. A quick search with *Mathematica* in the range $0 \leqslant n \leqslant 200$, $0 \leqslant m \leqslant 93$ and, with our conventions, we obtain all solutions listed in Theorem 1.1.

From now on we assume that $n > 200$. Further, from (3.6) we obtain $m > 91$ and also $n > m$. From the Binet formulas (3.1) we rewrite our equation as

$$c_1\gamma^n - d_1\alpha^m = d_2\beta^m + d_3\overline{\beta}^m - c_2\delta^n - c_3\overline{\delta}^n.$$

By taking absolute values, we get

$$|c_1\gamma^n - d_1\alpha^m| < 1,$$

from our assumption on m and n. Thus, dividing through by $d_1\alpha^m$ and from (3.5) we obtain

$$\left|\frac{c_1}{d_1}\gamma^n\alpha^{-m} - 1\right| < \frac{1}{d_1\alpha^m} \leqslant \frac{1}{d_1\alpha\gamma^{n-3}} < \frac{2}{\gamma^{n-3}} < \frac{1}{\gamma^{n-6}}. \tag{3.7}$$

Let Λ be the expression inside the absolute value. We claim that $\Lambda \neq 0$. To see this, we consider the $\mathbb{Q}$-automorphism σ of the Galois extension $\mathbb{Q}(\alpha, \beta, \gamma, \delta)$ of $\mathbb{Q}$ given by $\sigma(\alpha) = \alpha$, $\sigma(\beta) = \beta$, $\sigma(\gamma) = \delta$, $\sigma(\delta) = \alpha$. Thus, $\sigma(\overline{\beta}) = \overline{\beta}$ and $\sigma(\overline{\delta}) = \overline{\delta}$. If $\Lambda = 0$ then $\sigma(\Lambda) = 0$. Thus

$$c_2\delta^n = \sigma(c_1\gamma^n) = \sigma(d_1\alpha^m) = d_1\alpha^m.$$

By taking absolute values we get

$$1 > |c_2||\delta|^n = d_1\alpha^m > 1,$$

where the left-hand side inequality is because $1 > |c_2|, |\delta|$ and the right-hand side inequality is because $\alpha > 1$ and $m > 91$, which is absurd.

Now, we shall apply Matveev's inequality to Λ. To do this we consider the real number field $\mathbb{L} := \mathbb{Q}(\alpha, \gamma)$ which is of degree $d_{\mathbb{L}} = 9$ and

$$\alpha_1 = \frac{c_1}{d_1},\ \alpha_2 = \gamma,\ \alpha_3 = \alpha, \quad b_1 = 1,\ b_2 = n,\ b_3 = -m.$$

Thus we take $B = n$. Further, $h(\alpha_2) = \log\gamma/3$, and $h(\alpha_3) = \log\alpha/3$. For $h(\alpha_1)$ we use the properties of the height and conclude that

$$h(\alpha_1) \leqslant \log\gamma + \frac{5}{3}\log\alpha + 10\log 2.$$

Thus we take $A_1 = 74.1$, $A_2 = 0.85$, $A_3 = 1.9$. Now, from Theorem 2 we have

$$\log|\Lambda| > -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 9^2 \cdot (1 + \log 9) \cdot (1 + \log n) \cdot 74.1 \cdot 0.85 \cdot 1.9$$

which compared with (3.7) we get

$$(n - 6)\log\gamma < 4.43763 \times 10^{15}(2\log n),$$

where we use $1 + \log n < 2\log n$ since $n > 200$. Thus $n < 3.15623 \times 10^{16}\log n$ and, from Lemma 2.2 we conclude that

$$n < 2.39815 \times 10^{18}. \tag{3.8}$$

Now we reduce this upper bound on n. To do this, let Γ be defined as

$$\Gamma = n\log\gamma - m\log\alpha + \log\frac{c_1}{d_1},$$

and we go to (3.7). Observe that $e^\Gamma - 1 = \Lambda \neq 0$. Thus $\Gamma \neq 0$. If $\Gamma > 0$ we obtain

$$0 < \Gamma < e^\Gamma - 1 = \Lambda = |\Lambda| < \frac{1}{\gamma^{n-6}}.$$

If $\Gamma < 0$, we then have $1 - e^{\Gamma} = |e^{\Gamma} - 1| = |\Lambda| < 1/2$ since $n > 200$. This implies that $e^{|\Gamma|} < 2$. Thus,

$$0 < |\Gamma| < e^{|\Gamma|} - 1 = e^{|\Gamma|}|\Lambda| < \frac{2}{\gamma^{n-6}}.$$

So, in both cases we have

$$0 < |\Gamma| < \frac{2}{\gamma^{n-6}}.$$

Dividing through by $\log \alpha$ we obtain

$$0 < |n\tau - m + \mu| < \frac{2\gamma^6}{\log \alpha}\frac{1}{\gamma^n} < \frac{18}{\gamma^n} \tag{3.9}$$

where

$$\tau := \frac{\log \gamma}{\log \alpha}, \quad \mu := \frac{\log(c_1/d_1)}{\log \alpha}.$$

Now we will apply Lemma 2.1. To do this, we take $M := 2.39815 \times 10^{18}$ which is the upper bound on n by (3.8). With the help of *Mathematica* we found that the convergent

$$\frac{p_{31}}{q_{31}} = \frac{6879714423060542181}{14908790976189525844}$$

of τ is such that $q_{31} > 6\,M$ and $\varepsilon = \|q_{31}\,\mu\| - M\,\|q_{31}\,\tau\| = 0.400051 > 0$. Thus, from Lemma 2.1 with $A := 18$, $B := \gamma$ we get

$$n < \frac{\log(18\,q_{31}/\varepsilon)}{\log \gamma} < 171,$$

which contradicts our assumption on n. This completes the proof of Theorem 1.1.

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