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A note on P-I-convergence

Una nota sobre P-I-convergencia

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Abstract

In this article, we use the notions of pre-open and pre-I-open sets to introduce the idea of pre-I-convergence which we will denoted by P-I-convergence, we also show some of its properties. Besides, some basic properties of pre-I-Fréchet-Urysohn space is shown. Moreover, notions related to pre-I-sequential and pre-I-sequentially are proved. Furthermore, we show some relations of pre-I-irresolute functions between preserving pre-I-convergence functions and pre-I-covering functions.

Key words and phrases: pre-I-convergence, pre-I-irresolute functions, preserving pre-I-convergence functions, pre-I-sequentially open, pre-I-sequential spaces, pre-I-covering functions, pre-I-Fréchet-Urysohn spaces.

Resumen

En este artículo, usamos las nociones de conjuntos pre-abierto y pre-I-abierto para introducir la idea de pre-I-convergencia la cual vamos a denotar por P-I-convergencia, también mostramos algunas de sus propiedades. Además, algunas propiedades básicas del espacio pre-I-Fréchet-Urysohn son mostradas. Adicionalmente, nociones relativas a espacios pre-I-secuenciales y pre-I-secuencialmente abiertos son probadas. Además, mostramos algunas relaciones entre funciones pre-I-irresolutas, funciones que preservan pre-I-convergencia y funciones de pre-I-cobertura.

Palabras y frases clave: pre-I-convergencia, funciones pre-I-irresolutas, funciones que preservan pre-I-convergencia, pre-I-secuencialmente abierto, espacios pre-I-secuenciales, funciones de pre-I-cobertura, espacios pre-I-Fréchet-Urysohn.

1 Introduction

The notion of ideal was introduced by Kuratowski in 1933 in [4], an ideal I on a space X is a collection of elements of X which satisfies: (1) $\emptyset \in I$; (2) If $A, B \in I$ then $A \cup B \in I$; and (3) if $B \subset I$ and $A \subset B$, then $A \in I$. This notion has been grown in several concepts of general topology. In 2019, Zhou and Lin in [7] used the notion of ideal on the set $\mathbb{N}$ to extend the notion of I-convergence, those results were useful for the developing of this paper. On the other hand, in 1982, Mahhour et al. in [6] introduced the concept of pre-open sets in a topological spaces, after that

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in 1999, Dontchev in [2] presented the idea of pre-I-open sets and pre-I-continuous functions in ideal topological spaces. In this article, we took whole the notions mentioned above and we define other properties on pre-I-convergence and we study the relation between pre-I-sequentially open and pre-I-sequential space. Moreover, we define and study some basic properties of preserving pre-I-convergence functions and pre-I-covering functions, furthermore we prove some relations with pre-I-irresolute functions. Besides, the idea of pre-I-Fréchet-Urysohn spaces is defined.

Throughout this paper, the terms (X, τ) and (Y, σ) means topological spaces on which no separation axioms are assumed unless otherwise mentioned. Additionally, we sometimes write X or Y instead of (X, τ) or (Y, σ), respectively. By other hand, $|A|$ will denote the cardinality of set A.

2 pre-I-convergence

We first introduce some definitions.

Definition 2.1. Let (X, τ) be a topological space, $A \subset X$ and $x \in X$. Then, A is said to be pre-neighbourhood of x if and only if there exits a a pre-open set B such that $x \in B \subset A$.

Definition 2.2. An ideal $I \subseteq \mathbb{N}$ is said to be non-trivial, if $I = \varnothing$ and $I \neq \mathbb{N}$. A non-trivial ideal $I \subseteq K$ is called admissible if $I \supseteq \{\{n\} : n \in \mathbb{N}\}$.

Definition 2.3. Let I be an ideal on $\mathbb{N}$ and X be a topological space. A sequence $(x_n)_{n\in\mathbb{N}}$ is called pre-I-convergent to a point $x \in X$, provided for any pre-neighbourhood V of x, it has $A_V = \{n \in \mathbb{N} : x_n \notin V\} \in I$, which is denoted by p-I-$\lim_{n\to\infty} x_n = x$ or $x_n \to^{pI} x$ and the point x is called the p-I-limit of the sequence $(x_n)_{n\in\mathbb{N}}$.

Definition 2.4. Let (X, τ) be a topological space and $A \subset X$. Then, A is called pre-I-sequentially open if and only if no sequence in $X - A$ has a pre-I-limit in A. i.e. sequence can not pre-I-converge out of a pre-I-sequentially closed set.

Definition 2.5. Let I be an ideal on $\mathbb{N}$ and X be a topological space, then

1. A subset J of X is said to be pre-I-closed if for each sequence $(x_n)_{n\in\mathbb{N}} \subseteq J$ with $x_n \to^{pI} x \in X$, then $x \in J$.
2. A subset V of X is said to be pre-I-open if $X - V$ is pre-I-closed.
3. X is said to be a pre-I-sequential space if each pre-I-closed set in X is closed.

Definition 2.6. Let (X, τ) be a topological space. Then, X is pre-I-sequential when any set A is pre-open if and only if it is pre-I-sequentially open.

Now, we show some results.

Lemma 2.1 (cf. [7])**.** *Let I be an ideal on $\mathbb{N}$ and X be a topological space. If a sequence $(x_n)_{n\in\mathbb{N}}$ I-converges to a point $x \in X$ and $(y_n)_{n\in\mathbb{N}}$ is a sequence in X with $\{n \in \mathbb{N} : x_n \neq y_n\} \in I$, then the sequence $(y_n)_{n\in\mathbb{N}}$ I-converges to $x \in X$*

Lemma 2.2 (cf. [7])**.** *Let $I \subseteq J$ be two ideals of $\mathbb{N}$. If $(x_n)_{n\in\mathbb{N}}$ is a sequence in a topological space X such that $x_n \to^I x$, then $x_n \to^J x$.*

Lemma 2.3. *Let (X, τ) be a topological space. Then, $B \subset X$ is pre-I-sequentially open if and only if every sequence with pre-I-limit in B has all but finitely many terms in B. Where the index set of the part in B of the sequence does not belong to I.*

Proof. Suppose that B is not a pre-I-sequentially open, then there is a sequence with terms in $X - B$, but pre-I-limit in B. Conversely, suppose that $(x_n)_{n\in\mathbb{N}}$ is a sequence with infinitely many terms in $X - B$ such that pre-I-converges to $y \in B$ and the index set of the part in B of the sequence does not belong to I. Then, $(x_n)_{n\in\mathbb{N}}$ has a subsequence in $X - B$ that has to still converges to $y \in B$ and so B is not pre-I-sequentially open. □

Lemma 2.4. *Let I and J be two ideals of $\mathbb{N}$ where $I \subseteq J$ and X is a topological space. If $V \subseteq X$ is pre-J-open, then it is pre-I-open.*

Proof. Let $V \subseteq X$ be pre-I-open. Then, $X - V$ is pre-I-closed set, so every sequence $(x_n)_{n\in\mathbb{N}}$ in $X - V$ with $x_n \to^{pI} x$, hold that $x_n \to^{pJ} x$, by Lemma 2.2. So, $x \in X - V$ and therefore, V is pre-J-open. □

Corollary 2.1. *Let I and J be two ideals of $\mathbb{N}$, where $I \subseteq J$. If a topological space X is pre-I-sequential, then it is pre-J-sequential.*

Lemma 2.5. *Let I be an ideal on $\mathbb{N}$ and X be a topological space. If a sequence $(x_n)_{n\in\mathbb{N}}$ pre-I-convergent to a point $x \in X$ and $(y_n)_{n\in\mathbb{N}}$ is a sequence in X with $\{n \in \mathbb{N} : x_n \neq y_n\} \in I$, then the sequence $(y_n)_{n\in\mathbb{N}}$ pre-I-convergent to $x \in X$.*

Proof. The proof is followed by the Lemma 2.1 and Definition 2.3. □

Lemma 2.6. *Let X be a topological space X, $A \subset X$ and I be an ideal on $\mathbb{N}$. Then, the following statements are equivalent.*

1. *A is pre-I-open.*
2. *$\{n \in \mathbb{N} : x_n \in A\} \notin I$ for each sequence $(x_n)_{n\in\mathbb{N}}$ in X with $x_n \to^{pI} x \in A$.*
3. *$|\{n \in \mathbb{N} : x_n \in A\}| = \theta$ for each sequence $(x_n)_{n\in\mathbb{N}}$ in X with $x_n \to^{pI} x \in A$.*

Proof. (1) ⇒ (2) : Suppose that A is a pre-I-open set of X and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X satisfying $x_n \to^{pI} x \in A$. Now, take $N_0 = \{n \in \mathbb{N} : x_n \in A\}$. If $N_0 \in I$, then $N_0 \neq \mathbb{N}$ and so $A \neq X$. Now, take a point $a \in X - A$ and define the sequence $(y_n)_{n\in\mathbb{N}}$ in X by

$$y_n = \begin{cases} a & \text{if} \quad n \in N_0 \\ x_n & \text{if} \quad n \notin N_0 \end{cases} .$$

By Lemma 2.5, the sequence $(y_n)_{n\in\mathbb{N}}$ pre-I-converges to x. We can see that $X - A$ is pre-I-closed and $(y_n)_{n\in\mathbb{N}} \subseteq X - A$, in consequence $x \in X - A$ and this is a contradiction. Therefore, $N_0 \notin I$.

The implication (2) ⇒ (3) it follows form the notion that the ideal I is admissible.

Now, it shows the following implication. (3) ⇒ (1) : Let A not be pre-I-open in X. Then, $X - A$ is no pre-I-closed and there is a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X - A$ with $x_n \to^{pI} x \in A$ and this is a contradiction. □

Theorem 2.1. *Every pre-I-sequential space is hereditary with respect to pre-I-open (pre-I-closed) subspaces.*

Proof. Let X be a pre-I-sequential space. Now, suppose that A is a pre-I-open set of X. Then, A is pre-open in X. Now, we can see that A is pre-I-sequential. Let V be a pre-I-open set in A, thus V is pre-open in X. Indeed, by the Definition 2.6, if we show that V is pre-I-open in X, it will be sufficient.

Now, suppose that there is a point $y \in X - V$ and take an arbitrary $x \in V$ and a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ with $x_n \to^{pI} x$ in X. Since, A is pre-open in X and $x \in A$, the set $\{n \in \mathbb{N} : x_n \notin A\} \in I$. We define the sequence $(y_n)_{n\in\mathbb{N}}$ in X by

$$y_n = \begin{cases} x_n & \text{if} \quad x_n \in A \\ y & \text{if} \quad x_n \notin A \end{cases}.$$

By the Lemma 2.5, the sequence $(y_n)_{n\in\mathbb{N}}$ pre-I-converges to x. Since $|\{n \in \mathbb{N} : x_n \notin V\}| = |\{n \in \mathbb{N} : y_n \notin V\}|$ and by the Lemma 2.6, V is pre-I-open in X.

Now, let A be a pre-I-closed set of X. Then, A is pre-closed in X. For any pre-I-closed set J of A. It has to show that J is pre-closed in X. Since X is a pre-I-sequential space, it is enough that J is pre-I-closed in X. Hence, let $(x_n)_{n\in\mathbb{N}}$ be an arbitrary sequence in J with $x_n \to^{pI} x \in X$. It obtains that $x \in J$. Indeed, since A is pre-closed, it has that $x \in A$ and so $x \in J$ since J is a pre-I-closed set of A. □

Theorem 2.2. *pre-I-sequential spaces are preserved by topological sums.*

Proof. Let $\{X_\delta\}_{\delta\in\Delta}$ be a family of pre-I-sequential spaces. Take $X = \bigoplus_{\delta\in\Delta} X_\delta$, being the topological sum of $\{X_\delta\}_{\delta\in\Delta}$. Now, it will show that the topological sum is a pre-I-sequential space. Let J be a pre-I-closed set in X. For each $\delta \in \Delta$, since X_δ is pre-closed in X, $J \cap X_\delta$ is pre-I-closed in X. We can see that $J \cap X_\delta \subseteq X_\delta$ and $J \cap X_\delta$ is pre-I-closed in X_δ. By the assumption, it has that $J \cap X_\delta$ is pre-closed in X_δ. By the definition of topological sums, it gets that J is pre-closed in X. Therefore, the topological sum X is a pre-I-sequential space. □

Remark 2.1. The union of a family of pre-I-open sets of a topological space is pre-I-open. Therefore, the intersection of finitely many sequentially pre-I-open sets is sequentially pre-I-open

Definition 2.7 (cf. [7])**.** Let I be an ideal on $\mathbb{N}$ and A be a subset of a topological space X. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is I-eventually in A if there is $B \in I$ such that, for all $n \in \mathbb{N} - B$, $x_n \in A$.

Proposition 2.1. *Let I be a maximal ideal on $\mathbb{N}$ and X be a topological space. Then, A is a subset of X where A is pre-I-open if and only if each pre-I-convergent sequence in X, converging to a point of A is I-eventually in A.*

Proof. Let A be a pre-I-open and $x_n \to^{pI} x \in A$. Since I is maximal, by the Lemma 2.6, $B = \{n \in \mathbb{N} : x_n \notin A\} \in I$. Therefore, for each $n \in \mathbb{N} - B, x_n \in A$, i.e., the sequence $(x_n)_{n\in\mathbb{N}}$ is I-eventually in A. □

Theorem 2.3. *Let I be a maximal ideal of $\mathbb{N}$ and X be a topological space. If V, W are two pre-I-open sets of X, then $V \cap W$ is pre-I-open.*

Proof. It will be shown that every pre-I-convergent sequence converging to a point in $V \cap W$ is I-eventually in it. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X such that $x_n \to^{pI} x \in V \cap W$. There are $A, S \in I$ such that for each $n \in \mathbb{N} - A, x_n \in V$ and for each $n \in \mathbb{N} - S, x_n \in W$. Since $A \cup S \in I$ and for each $n \in \mathbb{N} - (A \cup S), x_n \in V \cap W$, therefore $V \cap W$ is a pre-I-open set. □

3 Further properties

3.1 pre-I-irresolute functions

In this part, it is introduced pre-I-irresolute functions and it shows some relations among continuous and pre-I-continuous functions.

Definition 3.1. (cf. [1]). Let $f : (X, \tau) \to (Y, \sigma)$ be a functions. f is called sequentially continuous provided V is sequentially open in Y, then $f^{-1}(V)$ is sequentially open in X.

Definition 3.2. Let I be an ideal on $\mathbb{N}$, $(X, \tau), (Y, \sigma)$ be a topological spaces and $f : (X, \tau) \to (Y, \sigma)$ be a function, then.

1. f is said to be preserving pre-I-convergence provided for each sequences $(x_n)_{n\in\mathbb{N}}$ in X with $x_n \to^{pI} x$, the sequence $(f(x_n))_{n\in\mathbb{N}}$ pre-I-converges to $f(x)$.

2. f is said to be pre-I-irresolute if for each pre-I-open V in Y, then $f^{-1}(V)$ is pre-I-open in X (cf. [2]).

Lemma 3.1 (cf. [2])**.** *Every pre-I-irresolute function is pre-I-continuous.*

Theorem 3.1. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function. If f is continuous, then f preserves pre-I-convergence.*

Proof. Suppose that f is continuous and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X such that $x_n \to^{pI} x \in X$. Now, let V be an arbitrary semi-neighbourhood of $f(x)$ in Y. Since f is continuous, $f^{-1}(V)$ is a semi-neighbourhood of x. Therefore, it has that $\{n \in \mathbb{N} : x_n \notin f^{-1}(V)\} \in I$. We can see that $\{n \in \mathbb{N} : f(x_n) \notin V\} = \{n \in \mathbb{N} : x_n \notin f^{-1}(V)\}$. This implies that $\{n \in \mathbb{N} : f(x_n) \notin V\} \in I$. Hence, $f(x_n) \to^{pI} f(x)$. □

Theorem 3.2. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function. If f preserves pre-I-convergence, then f is pre-I-irresolute.*

Proof. Suppose that f preserves pre-I-convergence and J is an arbitrary pre-I-closed set in Y. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $f^{-1}(J)$ such that $x_n \to^{pI} x \in X$ .By the assumption, it has that $f(x_n) \to^{pI} f(x)$. Since $(f(x_n))_{n\in\mathbb{N}} \subseteq J$ and J is pre-I-closed in Y, hence $f(x) \in J$, i.e., $x \in f^{-1}(J)$. Therefore, $f^{-1}(J)$ is pre-I-closed in X and then f is pre-I-irresolute. □

Proposition 3.1. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function. If f preserves pre-I-convergence, then f is pre-I-continuous.*

Proof. The proof is followed by the Lemma 3.1 and Theorem 3.2. □

Theorem 3.3. *Let I be a maximal ideal on $\mathbb{N}$. Then, a function $f : (X, \tau) \to (Y, \sigma)$ is pre-I-irresolute if and only if it preserves pre-I-convergent sequences.*

Proof. Assume that f is pre-I-irresolute and a sequence $x_n \to^{pI} x$ in X. It has to show that $f(x_n) \to^{pI} f(x)$ in Y. Now, let V a semi-neighbourhood of $f(x)$. Then, $x \in f^{-1}(V)$ is pre-I-open in X, because V is pre-I-open in Y. Hence, there is $B \in I$ such that for all $n \in \mathbb{N} - B, x_n \in f^{-1}(V)$. This means that for all $n \in \mathbb{N} - B, f(x_n) \in V$. Therefore, $\{n \in \mathbb{N} : f(x_n) \notin V\} \in I$ and hence $f(x_n) \to^{pI} f(x)$. □

Theorem 3.4. *Let X be a pre-I-sequential space and $f : (X, \tau) \to (Y, \sigma)$ be a function. Then, the following statements are equivalent.*

1. *f is continuous.*
2. *f preserves pre-I-convergence.*
3. *f is pre-I-irresolute.*

Proof. (1) $\Leftrightarrow$ (2) was proved in the Theorems 3.1 and 3.2.

(3) $\Rightarrow$ (1) : Let f be pre-I-irresolute and J be an arbitrary semi-closed set in Y. Then, J is pre-I-closed in Y. Since f is pre-I-irresolute, $f^{-1}(J)$ is pre-I-closed in X. By assumption, it has that $f^{-1}(J)$ is semi-closed in X. Therefore, f is continuous. □

Proposition 3.2. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function and X be a pre-I-sequential space. Then, the following statements are equivalent.*

1. *f is continuous.*
2. *f is pre-I-continuous.*

Proof. The proof is followed by the Proposition 3.1 and Theorem 3.4. □

Lemma 3.2. *Let X be a pre-I-sequential space, then the function $f : (X, \tau) \to (Y, \sigma)$ is continuous if and only if it is sequentially continuous.*

Proof. Let X be a pre-I-sequential space, then every pre-I-closed set is closed, by [1] who proved that f is continuous if and only if f is sequentially continuous, indeed we have completed the proof. □

Corollary 3.1. *Let X be a pre-Isequential space and for a function $f : (X, \tau) \to (Y, \sigma)$ the following statements are equivalent.*

1. *f is continuous.*
2. *f preserves pre-I-convergence.*
3. *f is pre-I-continuous.*
4. *f is sequentially continuous.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) was proved in the Theorem 3.4, by the Lemma 3.2, we have (1) $\Leftrightarrow$ (4). □

3.2 pre-I-irresolute and pre-I-covering functions

Continuity and sequentially continuity are ones of the most important tools for studying sequential spaces on [5]. In this part, it is defined the concept of pre-I-covering functions and it is shown some of their properties.

Definition 3.3. (cf. [1]). Let $f : (X, \tau) \to (Y, \sigma)$ be a topological space. Then, f is said to be sequentially continuous provided $f^{-1}(V)$ is sequentially open in X, then V is sequentially open in Y.

Definition 3.4. (cf. [1]). Let $f : (X, \tau) \to (Y, \sigma)$ be a topological space. Then, f is said to be sequence-covering if, whenever $(y_n)_{n\in\mathbb{N}}$ is a sequence in Y covering to y in Y, there exits a sequence $(x_n)_{n\in\mathbb{N}}$ of points $x_n \in f^{-1}(y_n)$ for all $n \in \mathbb{N}$ and $x \in f^{-1}(y)$ such that $x_n \to x$.

Definition 3.5. Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then, f is said to be pre-I-covering if, whenever $(y_n)_{n\in\mathbb{N}}$ is a sequence in Y, pre-I-converging to y in Y, there exits a sequence $(x_n)_{n\in\mathbb{N}}$ of points $x_n \in f^{-1}(y_n)$ for all $n \in \mathbb{N}$ and $x \in f^{-1}(y)$ such that $x_n \to^{pI} x$.

Theorem 3.5. *Every pre-I-covering function is pre-I-irresolute.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a function and f be a pre-I-covering function. Now, assume that V is a non-pre-I-closed in Y. Then, there exits a sequence $(y_n)_{n\in\mathbb{N}} \subseteq V$ such that $y_n \to^{pI} y \notin V$. Since f is pre-I-covering, there exits a sequence $(x_n)_{n\in\mathbb{N}}$ of points $x_n \in f^{-1}(y_n)$ for all $n \in \mathbb{N}$ and $x \in f^{-1}(y)$ such that $x_n \to^{pI} x$. Now, we can see that $(x_n)_{n\in\mathbb{N}} \subseteq f^{-1}(V)$ and so $x \notin f^{-1}(V)$, therefore $f^{-1}(V)$ is non-pre-I-closed. In conclusion, f is pre-I-irresolute. □

Theorem 3.6. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then, the following statements hold.*

1. *If X is a pre-I-sequential space and f is continuous, then Y is a pre-I-sequential space and pre-I-irresolute.*

2. *If Y is a pre-Y-sequential space and f is pre-I-irresolute, then f is continuous.*

Proof.

1. Let X be a pre-I-sequential space and f be continuous. Suppose that V is pre-I-open in Y. Since f is a continuous function and X is a pre-I-sequential space, take an arbitrary sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ such that $x_n \to^{pI} x \in f^{-1}(V)$ in X. Since f is a continuous function, by the Theorem 3.1, $f(x_n) \to^{pI} f(x) \in V$. Now, since V is pre-I-open in Y and by the Lemma 2.6, it has that $|\{n \in \mathbb{N} : f(x_n) \in V\}| = \theta$, i.e., $|\{n \in \mathbb{N} : x_n \in f^{-1}(V)\}| = \theta$. Therefore, $f^{-1}(V)$ is pre-I-open in X.

   Now, assume that $V \subseteq Y$ such that $f^{-1}(V)$ is pre-I-open in X. Then, $f^{-1}(V)$ is a open set of X since X is pre-I-sequential space. as well know that f is continuous, then V is open in Y. Hence, f is continuous.

2. Let Y be a pre-I-sequential space and f be pre-I-irresolute. If $f^{-1}(V)$ is a open set of X, then $f^{-1}(V)$ is a pre-I-open set of X. Since f is pre-I-irresolute, V is a pre-I-open set of Y. Now, we know that Y is a pre-I-sequential space and so V is an open set of Y. Therefore, f is continuous.

□

By the Theorems 3.4 and 3.6 it is had the following result.

Corollary 3.2. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function, then f is continuous if and only if f is pre-I-irresolute and Y is a pre-I-sequential space.*

3.3 pre-I-Fréchet-Urysohn spaces

A topological space X is said to be Fréchet-Urysohn (cf. [3]) if for each $A \subseteq X$ and each $x \in Cl(A)$, there is a sequence in A converging to the point x in X. Now, in this part, it introduces the notion of pre-I-Fréchet-Urysohn and it shows a short result.

Definition 3.6. Let (X, τ) be a topological space. Then, X is said to be pre-I-Fréchet-Urysohn or simply P-I-FU, if for each $A \subseteq X$ and each $x \in pCl(A)$, there exits a sequence in A pre-I-converging to the point $x \in X$.

Lemma 3.3. *For two ideals I and J on $\mathbb{N}$ where $I \subseteq J$, if X is a P-I-FU-space, then it is a pre-J-FU-space.*

Proof. Let A be a subset of X and $x \in pCl(A)$. Since X is a P-I-FU-space, then there exits a sequence $(x_n)_{n\in\mathbb{N}}$ in A such that $x_n \to^{pI} x$, in consequence $x_n \to^{pI} x$ in X, and so X is pre-J-FU-space. □

Theorem 3.7. *Let (X, τ) be a topological space. Then, X is a P-I-FU-space, then X is a pre-I-sequential space.*

Proof. Let $\{A_\delta : \delta \in \Delta\}$ be a family of pre-I-closed subsets of X where $\delta \in \Delta \in X$, since X is a P-I-FU-space, by the Definition 3.6 $A_\delta \subseteq X$ and each $x \in pCl(A_\delta)$. Now, since A_δ is pre-I-closed $pCl(A_\delta) = A_\delta \in Cl(A)$, but by the Definition 3.6, there exits a pre-I-converging to the point $x \in pCl(A) \in Cl(A) \in X$, therefore $\{A_\delta : \delta \in \Delta\}$ is a closed set of X. In consequence X is a pre-I-sequential space. □

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