

On some interesting properties of p -laplacian equation

Sobre algunas propiedades interesantes de la ecuación p -laplaciana

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Abstract

In the present paper we establish, on the one hand, some singular solutions concerning to the 1-laplacian equation. On the other hand, we give some properties related to the weak solutions of p -laplacian equation

Key words and phrases: Singular solution, p -laplacian equation, p -harmonic function.

Resumen

En el presente artículo establecemos, por una parte, algunas soluciones singulares concernientes a la ecuación 1-laplaciana. Por otro lado, damos algunas propiedades relacionadas a la débil solución de la ecuación p -laplaciana.

Palabras y frases clave: Solución singular, ecuación p -laplaciana, función p -armónica.

1 Introduction

In this paper, we are investigating singular solutions and properties to the following equation which we shall call the p -Laplace equation [1-6, 9-11].

$$\operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0, \quad (1.1)$$

where p satisfies $1 \leq p \leq \infty$. The p -laplacian operator is defined as

$$\begin{aligned} \Delta_p u &= \operatorname{div}(|\nabla u|^{p-2} \nabla u) \\ &= |\nabla u|^{p-4} \left(|\nabla u|^2 \Delta u + (p-2) \sum_{i,j=1}^n \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j} \right) \end{aligned} \quad (1.2)$$

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There are several noteworthy values of p :

a) $p = \infty$. As $p \rightarrow \infty$ one encounters the infinity Laplacian equation

$$\sum_{i,j=1}^n \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j} = 0 \quad (1.3)$$

in \mathbb{R}^n , which some singular solutions are given by

$$a\sqrt{x_1^2 + \cdots + x_k^2} + b \quad (1 \leq k \leq n) \quad (1.4)$$

$$a_1 x_1 + \cdots + a_n x_n + b \quad (1.5)$$

$$a_1 x_1^{4/3} + \cdots + a_n x_n^{4/3} \quad \left(\sum_{j=1}^n a_j^3 = 0 \right) \quad (1.6)$$

b) $p = 2$. In this case we have the Laplace equation

$$\sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} = 0 \quad (1.7)$$

c) $p = 1$. In this case we obtain the 1-laplacian equation

$$\operatorname{div} \left(\frac{\nabla u}{|\nabla u|} \right) = 0 \quad (1.8)$$

For $x \in \mathbb{R}^3$ and under the assumption that $|\nabla u| \neq 0$, it then follows from (1.8)

$$\begin{aligned} |\nabla u|^2 \cdot \Delta u - \left\{ \left(\frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial z} \frac{\partial^2 u}{\partial x \partial z} \right) \frac{\partial u}{\partial x} + \right. \\ \left. + \left(\frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial y \partial x} + \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial z} \frac{\partial^2 u}{\partial y \partial z} \right) \frac{\partial u}{\partial y} + \right. \\ \left. + \left(\frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial z \partial x} + \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial z \partial y} + \frac{\partial u}{\partial z} \frac{\partial^2 u}{\partial z^2} \right) \frac{\partial u}{\partial z} \right\} = 0 \end{aligned} \quad (1.9)$$

The purpose of this paper is to obtain nontrivial singular solutions of (1.8) and some properties of weak solutions concerning the equation (1.1). The paper has been organized as follows: in section 2, we briefly review the basic definitions used in our subsequent discussions, next, the preliminary results are established in section 3. Section 4 present the main results.

2 Basic definitions

Here, we give some definitions used in our subsequent discussions. For more details, see [7-8, 12]

Definition 2.1. We denote by $C_0(\Omega)$ the space of all continuous functions on Ω with compact support.

Other interesting notations are

$$C_0^k(\Omega) = C^k(\Omega) \cap C_0(\Omega) \tag{2.1}$$

Definition 2.2. The Sobolev space $W^{1,p}(\Omega)$ is defined by

$$W^{1,p}(\Omega) = \left\{ u \in L^p \mid \exists g_1, g_2, \dots, g_N \in L^p \text{ such that } \int_{\Omega} u \frac{\partial \varphi}{\partial x_i} = - \int_{\Omega} g_i \varphi, \quad \forall \varphi \in C_0^\infty(\Omega), \quad \forall i = 1, 2, \dots, N \right\}$$

Definition 2.3. Let Ω be a domain in \mathbb{R}^n . We say that $u \in W_{loc}^{1,p}(\Omega)$ is a weak solution of the p -harmonic equation (1.1) in Ω , if

$$\int |\nabla u|^{p-2} \nabla u \cdot \nabla \eta dx = 0 \quad \forall \eta \in C_0^\infty(\Omega). \tag{2.2}$$

If, in addition, u is continuous, then we say u is a p -harmonic function.

Definition 2.4. Let Ω be a domain in \mathbb{R}^n . We say that $u \in W_{loc}^{1,p}(\Omega)$ is a classical solution of (1.1), if u satisfies (1.1)

Definition 2.5. We say that $u \in W_{loc}^{1,p}(\Omega)$ is a weak supersolution of the p -harmonic equation (1.1) in Ω , if

$$\int_{\Omega} |\nabla u|^{p-2} \cdot \nabla u \cdot \nabla \eta dx \geq 0 \tag{2.3}$$

for all nonnegative $\eta \in C_0^\infty(\Omega)$. For weak subsolution, the inequality is reversed.

Definition 2.6. Let $\Omega \subset \mathbb{R}^n$ be an open set, a linear differential operator of second order $L : C^2(\Omega) \rightarrow C(\Omega)$ is defined as

$$L(u) = - \sum_{j=1}^n \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial u}{\partial x_j} \right) + \sum_{i=1}^n a_i \frac{\partial u}{\partial x_i} + a_0 u \tag{2.4}$$

where $a_{ij} \in C^1(\Omega), a_i \in C(\Omega), \forall i, j = 1, \dots, n$ y $a_0 \in C(\bar{\Omega})$.

Definition 2.7. Let V an K -vectorial space. A function $g : V \times V \rightarrow K$ is called a *bilinear form* if

- i) $g(u + v, w) = g(u, w) + g(v, w), \quad \forall u, v, w \in V,$
- ii) $g(\lambda v, w) = \lambda g(v, w), \quad \forall \lambda \in K, \forall v, w \in V,$
- iii) $g(u, v + w) = g(u, v) + g(u, w), \quad \forall u, v, w \in V,$
- iv) $g(v, \lambda w) = \lambda g(v, w), \quad \forall \lambda \in K, \forall v, w \in V.$

Definition 2.8. Let H be a Hilbert space, we say that a bilinear form $g : H \times H \rightarrow \mathbb{R}$ is

- a) continuous if there exists a constant $C > 0$ such that

$$|g(u, v)| \leq C \|u\|_H \cdot \|v\|_H \quad \forall u, v \in H. \tag{2.5}$$

b) coercive if there exists a constant $\theta > 0$ such that

$$g(u, v) \geq \theta \|u\|_H^2 \quad \forall u \in H. \quad (2.6)$$

Definition 2.9. The bilinear form $g : H_0^1(\Omega) \times H_0^1(\Omega) \rightarrow \mathbb{R}$ connected with the operator L is defined as

$$g(u, v) = \sum_{i,j=1}^N \int_{\Omega} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{i=1}^N \int_{\Omega} a_i \frac{\partial u}{\partial x_i} \cdot v + \int_{\Omega} a_0 uv \quad (2.7)$$

for all $u, v \in H_0^1(\Omega)$.

3 Preliminary results

Theorem 3.1. *Let suppose that the bilinear form (2.7) is coercive with the constant of coersity θ . Then the bilinear form (2.7) is continuous and furthermore exist $\alpha, \gamma \geq 0$ such that [8]*

$$\alpha \cdot \|u\|_{H_0^1(\Omega)}^2 \leq g(u, u) + \gamma \cdot \|u\|_{L^2(\Omega)}^2 \quad (3.1)$$

for all $u \in H_0^1(\Omega)$. Where

$$\alpha = \frac{\theta}{2} \quad \gamma = \frac{1}{4\varepsilon} \sum_{i=1}^n \|a_i\|_{L^\infty(\Omega)} + \|a_0\|_{L^\infty(\Omega)}. \quad (3.2)$$

Theorem 3.2. (Caccioppoli) *If u is a weak solution of (1.1) in Ω , then*

$$\int_{\Omega} \xi^p \cdot |\nabla u|^p dx \leq p^p \cdot \int_{\Omega} |u|^p \cdot |\nabla \xi|^p dx \quad (3.3)$$

for al $\xi \in C_c^\infty(\Omega), \xi \geq 0$.

Proof. Use

$$\begin{aligned} \eta &= \xi^p u \\ \nabla \eta &= \xi^p \nabla u + p \xi^{p-1} u \nabla \xi \end{aligned}$$

By the equation (2.2) and Holder's inequality

$$\begin{aligned} \int_{\Omega} \xi^p |\nabla u|^p dx &= -p \int_{\Omega} \xi^{p-1} \cdot u \langle |\nabla u|^{p-2} \nabla u, \nabla \xi \rangle dx \\ &\leq p \int_{\Omega} |\xi \cdot \nabla u|^{p-1} \cdot |u \cdot \nabla \xi| dx \\ &\leq p \left\{ \int_{\Omega} \xi^p |\nabla u|^p dx \right\}^{1-\frac{1}{p}} \cdot \left\{ \int_{\Omega} |u|^p \cdot |\nabla \xi|^p \right\}^{\frac{1}{p}} \end{aligned}$$

The estimate follows. □

Theorem 3.3. *If $v > 0$ is a weak supersolution of (1.1) in Ω , then*

$$\int_{\Omega} \xi^p \cdot |\nabla \log v|^p dx \leq \left(\frac{p}{p-1}\right)^p \cdot \int_{\Omega} |\nabla \xi|^p dx \tag{3.4}$$

whenever $\xi \in C_c^\infty(\Omega), \xi \geq 0$.

Proof. One may add constants to the weak supersolutions. First, prove the estimate for $v(x) + \epsilon$ in place of $v(x)$. Then let $\epsilon \rightarrow 0$ in

$$\int_{\Omega} \frac{\xi^p \cdot |\nabla v|^p}{(v + \epsilon)^p} dx \leq \left(\frac{p}{p-1}\right)^p \cdot \int_{\Omega} |\nabla \xi|^p dx \tag{3.5}$$

Hence we may assume that $v(x) \geq \epsilon > 0$. Next use the test function $\eta = \xi^p v^{1-p}$. Then

$$\nabla \eta = p \xi^{p-1} v^{1-p} \nabla \xi - (p-1) \xi^p v^{-p} \nabla v \tag{3.6}$$

and we obtain

$$\begin{aligned} (p-1) \int_{\Omega} \xi^p v^{-p} |\nabla v|^p dx &\leq p \int_{\Omega} \xi^{p-1} \cdot v^{1-p} \langle |\nabla v|^{p-2} \nabla v, \nabla \xi \rangle dx \\ &\leq p \int_{\Omega} \xi^{p-1} \cdot v^{1-p} \cdot |\nabla v|^{p-1} \cdot |\nabla \xi| dx \\ &\leq p \left\{ \int_{\Omega} \xi^p \cdot v^{-p} |\nabla v|^p dx \right\}^{1-\frac{1}{p}} \cdot \left\{ \int_{\Omega} |\nabla \xi|^p dx \right\}^{\frac{1}{p}} \end{aligned}$$

from which the result follows. □

Theorem 3.4. *Suppose that $1 \leq p \leq \infty$ and Ω is a bounded open set. Then exists a constant C (depending on Ω and p) such that [7]*

$$\|u\|_{L^p} \leq C \|\nabla u\|_{L^p}, \quad \forall u \in W_0^{1,p}(\Omega) (1 \leq p \leq \infty)$$

In particular, the expression $\|\nabla u\|_{L^p}$ is a norm on $W_0^{1,p}(\Omega)$, and it is equivalent to the norm $\|u\|_{W^{1,p}}$

4 Main Results

4.1 Some results of weak solutions

Theorem 4.1. *A C^2 function u that satisfies (1.1) is a weak solution of (1.1)*

Proof. Multiply (1.1) by $\eta \in C_0^\infty(\Omega)$ and integrate by parts; we obtain

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \eta dx = 0 \quad \forall \eta \in C_0^\infty(\Omega)$$

as required. □

Theorem 4.2. *Let $u \in W^{1,p}(\Omega)$ be a weak solution of p -harmonic equation (1.1) in Ω , then*

$$\|\xi \cdot |\nabla u|\|_{L^p} \leq p(\|u\|_{L^{2p}}^2 + \|\nabla \xi\|_{L^{2p}}^2) \quad \forall \xi \in C_0^\infty(\Omega), \xi \geq 0.$$

Proof. From Theorem 3.2, we have

$$\int_{\Omega} \xi^p \cdot |\nabla u|^p dx \leq p^p \cdot \int_{\Omega} |u|^p \cdot |\nabla \xi|^p dx \quad (4.1)$$

In terms of (4.1), it then follows that

$$\|\xi \cdot |\nabla u|\|_{L^p} \leq p \cdot \| |u| \cdot |\nabla \xi| \|_{L^p} \quad (4.2)$$

To continue we need the Young inequality

$$a \cdot b \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2, \quad a, b \in \mathbb{R}, \varepsilon \in \mathbb{R}_+ \quad (4.3)$$

We obtain

$$\begin{aligned} \|\xi \cdot |\nabla u|\|_{L^p} &\leq p \| |u| \cdot |\nabla \xi| \|_{L^p} \\ &\leq p \left[\varepsilon \| |u|^2 \|_{L^p} + \frac{1}{4\varepsilon} \| |\nabla \xi|^2 \|_{L^p} \right] \\ &\leq p \left(\varepsilon \| |u|^2 \|_{L^p} + \frac{1}{4\varepsilon} \| |\nabla \xi|^2 \|_{L^p} \right) \end{aligned} \quad (4.4)$$

We insert $\varepsilon \in [\frac{1}{4}, 1]$ into the inequality (4.4). This yields

$$\begin{aligned} \|\xi \cdot |\nabla u|\|_{L^p} &< p \left[\varepsilon \| |u|^2 \|_{L^p} + (1 - \varepsilon) \| |u|^2 \|_{L^p} \right. \\ &\quad \left. + \frac{1}{4\varepsilon} \| |\nabla \xi|^2 \|_{L^p} + \frac{4\varepsilon - 1}{4\varepsilon} \| |\nabla \xi|^2 \|_{L^p} \right] \\ &= p \left[\| |u|^2 \|_{L^p} + \| |\nabla \xi|^2 \|_{L^p} \right] \\ &= p(\|u\|_{L^{2p}}^2 + \|\nabla \xi\|_{L^{2p}}^2), \end{aligned}$$

as required. \square

Theorem 4.3. *Let suppose that the constant of coersity of the bilinear form (2.7) is θ . Let also $v \in W^{1,p}(\Omega)$ be a positive ($v > 0$) weak supersolution of the p -harmonic equation (1.1) in Ω , then there exist constants $\beta > 0$ and $\gamma \geq 0$ such that*

$$\beta \|u \cdot |\nabla \log v|\|_{L^2(\Omega)}^2 \leq g(u, u) + \gamma \cdot \|u\|_{L^2(\Omega)}^2 \quad (4.5)$$

for all $u \in C_0^\infty(\Omega), u \geq 0$, with γ and $g(u, u)$ given by (3.2) and (2.7) respectively.

Proof. Let $p = 2$. By Theorem 3.3, we obtain inequality

$$\|u \cdot |\nabla \log v|\|_{L^2(\Omega)}^2 \leq 4 \cdot \|\nabla u\|_{L^2(\Omega)}^2$$

which can be written as

$$\|u \cdot |\nabla \log v|\|_{L^2(\Omega)}^2 \leq 4 \cdot \|\nabla u\|_{L^2(\Omega)}^2 = 4 \cdot \|u\|_{H_0^1(\Omega)}^2 \quad (4.6)$$

Combining the inequality (3.1) with the estimate (4.6), we obtain

$$\alpha \|u \cdot |\nabla \log v|\|_{L^2(\Omega)}^2 \leq 4 \cdot g(u, u) + 4 \cdot \gamma \cdot \|u\|_{L^2(\Omega)}^2 \quad (4.7)$$

Divide out the common factor. We arrive at

$$\beta \|u \cdot |\nabla \log v|\|_{L^2(\Omega)}^2 \leq g(u, u) + \gamma \cdot \|u\|_{L^2(\Omega)}^2, \quad (4.8)$$

with the constant $\beta = \frac{\alpha}{4}$. This concludes the proof. \square

4.2 Singular solutions of 1-Laplacian equation

The purpose of this section is to prove that the functions

$$u = (x_1 + x_2 + \cdots + x_n + d_0)e^{x_1+x_2+\cdots+x_n+d_1} + d_2 \quad (4.9)$$

$$u = e^{a_1x_1+a_2x_2+\cdots+a_nx_n+d} + d_0 \quad (4.10)$$

$$u = \ln(a_1x_1 + a_2x_2 + \cdots + a_nx_n + d) + d_0 \quad (4.11)$$

$$u = \ln(e^{a_1x_1+a_2x_2+\cdots+a_nx_n+d_0} + d_1) + d_2 \quad (4.12)$$

$$u = a_1x_1 + a_2x_2 + \cdots + a_nx_n + d_0 + e^{a_1x_1+a_2x_2+\cdots+a_nx_n+d_1} \quad (4.13)$$

$$u = a_1x_1 + a_2x_2 + \cdots + a_nx_n + d_0 + \ln(a_1x_1 + a_2x_2 + \cdots + a_nx_n + d_1) \quad (4.14)$$

$$u = a_1x_1 + a_2x_2 + \cdots + a_nx_n + d_0 + \ln(e^{a_1x_1+a_2x_2+\cdots+a_nx_n+d_2} + d_1) \quad (4.15)$$

$$u = e^{a_1x_1+a_2x_2+\cdots+a_nx_n+d_0} + \ln(a_1x_1 + a_2x_2 + \cdots + a_nx_n + d_1) + d_2 \quad (4.16)$$

are singular solutions of the equation (1.8). Where $a_i (i = 0, \dots, n), a, b, c, d, d_0, d_1, d_2, d_3$ are real numbers. For $x \in \mathbb{R}^3$, we shall have

$$u = (x + y + z + d_0)e^{x+y+z+d_1} + d_2 \quad (4.17)$$

$$u = e^{ax+by+cz+d} + d_0 \quad (4.18)$$

$$u = \ln(ax + by + cz + d) + d_0 \quad (4.19)$$

$$u = \ln(e^{ax+by+cz+d_0} + d_1) + d_2 \quad (4.20)$$

$$u = ax + by + cz + d_0 + e^{ax+by+cz+d_1} \quad (4.21)$$

$$u = ax + by + cz + d_0 + \ln(ax + by + cz + d_1) \quad (4.22)$$

$$u = ax + by + cz + d_0 + \ln(e^{ax+by+cz+d_2} + d_1) \quad (4.23)$$

$$u = e^{ax+by+cz+d_0} + \ln(ax + by + cz + d_1) + d_2 \quad (4.24)$$

Theorem 4.4. *The function (4.17) is singular solution of the equation (1.9)*

Proof. Noticing that

$$|\nabla u|^2 \cdot \Delta u = 9 \cdot (x + y + z + d_0 + 1)^2 \cdot (x + y + z + d_0 + 2) \cdot e^{3(x+y+z+d_1)} \quad (4.25)$$

$$\left(\frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial z} \frac{\partial^2 u}{\partial x \partial z} \right) \frac{\partial u}{\partial x} = 3 \cdot (x + y + z + d_0 + 1)^2 \cdot (x + y + z + d_0 + 2) \cdot e^{3(x+y+z+d_1)} \quad (4.26)$$

$$\left(\frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial y \partial x} + \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial z} \frac{\partial^2 u}{\partial y \partial z} \right) \frac{\partial u}{\partial y} = 3 \cdot (x + y + z + d_0 + 1)^2 \cdot (x + y + z + d_0 + 2) \cdot e^{3(x+y+z+d_1)} \quad (4.27)$$

$$\left(\frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial z \partial x} + \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial z \partial y} + \frac{\partial u}{\partial z} \frac{\partial^2 u}{\partial z^2} \right) \frac{\partial u}{\partial z} = 3 \cdot (x + y + z + d_0 + 1)^2 \cdot (x + y + z + d_0 + 2) \cdot e^{3(x+y+z+d_1)} \quad (4.28)$$

and inserting (4.25)-(4.28) into the equation (1.9), we obtain

$$0 \cdot (x + y + z + d_0 + 1)^2 \cdot (x + y + z + d_0 + 2) \cdot e^{3(x+y+z+d_1)} = 0, \quad (4.29)$$

from which the proof follows. \square

Similar to the proof of the foregoing theorem, we have

Theorem 4.5. *Functions (4.18)-(4.24) are singular solutions of the equation (1.9).*

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