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Topological solitons II: CP^1 σ-model

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Abstract

We continue our study of classical topological solitons in nonlinear sigma models, focusing on the stability and scattering properties of the CP^1 model on the plane.

Key words: soliton, skyrmion, topology, scattering.

Solitones topológicos II: modelo-σ CP^1

Resumen

Continuamos nuestro estudio de solitones topológicos clásicos en modelos sigma no lineales, concentrándonos en las propiedades de estabilidad y dispersión del modelo CP^1 en el plano.

Palabras claves: solitón, skyrmión, topología, dispersión.

Introduction

In our previous paper (1) we presented an overview of topological solitons in one, two and three spatial dimensions. Based on that work, the present follow-up article pays specific attention to the popular nonlinear CP^1 sigma model in (2+1) dimension (two space, one time) both in its pure and Skyrme versions, and considers its stability and scattering properties. We limit ourselves to one and two solitons.

Physical and mathematical systems defined on the plane are the subject of much active research, covering topics that include Heisenberg ferromagnets, the quantum Hall effect, superconductivity, nematic crystals, topological fluids, vortices and solitary waves (2). Most of these systems are nonlinear. One of the simplest models in (2+1) dimensions which is both Lorentz covariant and which possesses soliton solutions is the nonlinear CP^1 or *O(3)* sigma model.

Among other things, sigma models are very useful as low dimensional analogues of important field theories in higher dimensions. For instance, the planar sigma CP^1 model exhibits conformal invariance, spontaneous symmetry breaking, asymptotic freedom and topological solitons, properties that resemble some of those present in a number of forefront field theories in (3+1) dimensions. An example of the latter is the Skyrme model of nuclear physics (3). Initially proposed as a theory of strong interactions between hadrons, it is now regarded as a low energy limit of quantum chromodynamics (4). The Skyrme scheme assumes that its topological solutions (skyrmions) correspond to ground states of light nuclei with the topological charge representing the baryon number. Of course, to compare with the properties of real nuclei one has to insert various quantum corrections to the classical results.

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Planar analogues of the Skyrme model involve the addition of extra terms to the original CP^1 lagrangian in order to stabilise the field solutions. Without such terms the invariance of the pure, planar CP^1 theory under dilations leads to the instability of its soliton-lumps. In the traditional approach one adds two terms: a Skyrme-like term which controls the shrinking of the lumps and a potential-like term which controls their expansion. Properly implemented, this procedure yields stable solitons as confirmed by numerical simulations (5).

In the following section we define the standard planar nonlinear *O(3)* or CP^1 model. In section 4 we present our Syrme version of the theory and in section 5 we lay out the numerical procedure. After presenting some basic results for one and two solitons, sections 6 and 7, respectively, we close our paper with some concluding remarks.

The *O(3)* model

One of the simplest Lorentz-invariant models in (2+1) dimensions is the nonlinear *O(3)* sigma model. It involves three real scalar fields $\phi(x^\mu) \equiv \left\{\phi_a(x^\mu), a = 1,2,3\right\}$ with the constraint that $\forall x^\mu \equiv (x^0, x^1, x^2) = (t, x, y)$ [speed of light set equal to unity] the fields lie on the unit sphere $S_2^{(\Phi)}$.

$$\vec{\phi}\cdot\vec{\phi} = 1 \qquad [1]$$

Subject to this constraint the lagrangian density reads

$$\mathcal{L}_0 = \tfrac{1}{4}\sum_{a=1}^{3}\sum_{\mu=0}^{2}\partial_\mu\phi_a\partial^\mu\phi_a = \tfrac{1}{4}(\partial_\mu\vec{\phi})\cdot(\partial^\mu\vec{\phi}), \qquad [2]$$

which is invariant under global *O(3)* rotations in internal space. Through the Euler-Lagrange equations with multipliers we find that the dynamics of the *O(3)* fields is governed by

$$\partial^\mu\partial_\mu\vec{\phi} - (\vec{\phi}\cdot\partial_\mu\partial^\mu\vec{\phi})\vec{\phi} = \vec{0}, \qquad [3]$$

which for the static case reduces to

$$\nabla^2\vec{\phi} - (\vec{\phi}\cdot\nabla^2\vec{\phi})\vec{\phi} = \vec{0}. \qquad [4]$$

Were it not for the constraint imposed on $\vec{\phi}$, the second term on the left-hand-side of the above equations would not be present, and the static non-singular solutions would be trivial. The condition [1] leads to finite-energy non-singular solutions: solitons. Furthermore, the interaction of the system is purely geometrical and defined by equation [1] which determines the curvature of the internal space . This is a particularity of chiral or sigma models.

It is straightforward to see that the kinetic and potential energies are given by

$$K = \tfrac{1}{4}\int(\partial_t\vec{\phi})\cdot(\partial_t\vec{\phi})dxdy, \qquad [5]$$

$$V = \tfrac{1}{4}\int(\partial_i\phi)\cdot(\partial_i\vec{\phi})\,dxdy \quad [i = 1,2]$$
$$= \tfrac{1}{4}\int(\nabla\vec{\phi}\cdot\nabla\vec{\phi})\,rdrd\theta, \qquad [6]$$

where $\nabla\phi = (\partial_r\vec{\phi}, \frac{1}{r}\partial_\theta)$.

The problem is completely specified by giving the boundary conditions. We take

$$\lim_{r\to\infty}\vec{\phi}(r,\theta) = \vec{\phi}^{(0)}, \quad \forall t, \qquad [7]$$

where the unit vector $\vec{\phi}^{(0)}$ is independent of the polar angle θ. This condition ensures a finite potential energy. In effect, finiteness of the energy demands

$$\lim_{r\to\infty}\sqrt{(r\partial_r\vec{\phi})^2 + (\partial_\theta\vec{\phi})^2} \to 0,$$
[8]

which implies [7].

It is interesting to note that the classical vacua ought to be represented by $\vec{\phi}^{(0)}$ for all $x \equiv (x, y)$. Since $\vec{\phi}^{(0)}$ can point in any direction, there is a continuous family of zero-energy solutions connected by *O(3)* rotations in internal space. This is an example of spontaneous symmetry breaking.

The boundary condition [7] defines a one-point compactification of the plane $\Re_2$, allowing us to consider $\vec{\phi}$ on the extended plane $\Re_2 \cup \{\infty\}$, topologically equivalent to $S_2^{(x)}$ (the superscript indicating that the sphere refers to compactified plane). Consequently, the field configurations we want are maps $S_2^{(x)} \to S_2^{(\Phi)}$ that can be labelled by an integral topological index Q. As sketched in (1), an expression for this index is obtained by pulling back the differential form

$$\begin{aligned} w &= \phi \cdot dS^{(\Phi)} \\ &= (\phi_1, \phi_2, \phi_3) \cdot (d\phi_2 \wedge d\phi_3, d\phi_3 \wedge d\phi_1, d\phi_1 \wedge d\phi_2) \end{aligned} \quad [9]$$

from the internal sphere to the 'physical' sphere. Using coordinates (x, y) in the latter, expansion of w yields

$$w = \begin{vmatrix} \phi_1 & \partial_x\phi_1 & \partial_y\phi_1 \\ \phi_2 & \partial_x\phi_2 & \partial_y\phi_2 \\ \phi_3 & \partial_x\phi_3 & \partial_y\phi_3 \end{vmatrix} dx \wedge dy \quad [10]$$

Relaxing the wedge notation we get

$$Q = \tfrac{1}{4\pi}\int_{S_2^{(x)}} \vec{\phi} \cdot (\partial_x\vec{\phi} \times \partial_y\vec{\phi})\, dxdy \quad [11]$$

quantity sometimes called the winding number because it gives the number of times that $\vec{\phi}$ ranges over the internal sphere as (x, y) ranges over the compactified plane once. The constant $1/4\pi$ normalises Q to an integer. Note that [9] is nothing but the element of area of the unit sphere $S_2^{(\Phi)}$; indeed, upon expanding ω in terms of local space polar coordinates (ϑ, φ) in internal space and parametrising

$$\vec{\phi} = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta),$$

we find the all familiar $\omega = \sin\vartheta\, d\vartheta\, d\varphi$. The topological charge now stems from

$$\int_{S_2^{(x)}} \phi w = Q \int_{S_2^{(\Phi)}} w,$$

hence equation [11]. Note as well that Q may be considered as the zero component of the topological current

$$k^{\mu} = \varepsilon^{\mu\nu\xi}\varepsilon^{abc}\phi_a\partial_\nu\phi_b\partial_\xi\phi_c, \quad [12]$$

where ε^{jkl} is the familiar Levi-Civita pseudotensor.

CP^1 formulation

In this formulation the soliton fields adopt a simpler form involving just one independent complex field, W, related to the fields $\vec{\phi}$ via the stereographic projection

$$W = \frac{\phi_1 + i\phi_2}{1 - \phi_3}. \quad [13]$$

Introducing complex coordinates $z = x + iy$ and $\bar{z} = x - iy$ on the extended plane and using the handy notation $\partial_z W = W_z, \partial_z(\partial_{\bar{z}} W) = W_{z\bar{z}}$ etc., the equation of motion [4] becomes

$$W_{z\bar{z}} = \frac{2\overline{W} W_z W_z}{|W|^2 + 1} = 0 \quad [14]$$

being $\overline{W}$ the complex conjugate of W.

In terms of W the potential energy and the topological index read

$$V = 2\int_{S_2^{(x)}} \frac{|W_z|^2 - |W_{\bar{z}}|^2}{\left(|W|^2 + 1\right)^2} dxdy, \quad [15]$$

$$Q = \tfrac{1}{\pi}\int_{S_2^{(x)}} \frac{|W_z|^2 - |W_{\bar{z}}|^2}{\left(|W|^2 + 1\right)^2} dxdy. \quad [16]$$

Clearly

$$V = \begin{cases} 2\pi Q + 4\int_{S_2^{(x)}} \dfrac{|W_z|^2}{|W|^2 + 1} dxdy \\ 2\pi[-Q] + 4\int_{S_2^{(x)}} \dfrac{|W_z|^2}{|W|^2 + 1} dxdy \end{cases}$$

i.e.

$$V \geq 2\pi|Q|. \quad [17]$$

The static solitons or instanton solutions correspond to the equality in [17]: solutions with $Q > 0$ (instantons) and solutions with $Q < 0$ (anti-instantons) obey, respectively,

$$W_{\bar{z}} = 0, \quad W_z = 0 \qquad [18]$$

which are nothing but the Cauchy-Riemann conditions for W being an analytic function of z or $\bar{z}$. Therefore, the general static solution of the planar CP^1 model is

$$W(z) = \lambda \prod_{j=1}^{k} \frac{z - a_j}{z - b_j} \qquad [19]$$

and its complex conjugate $W(\bar{z})$; λ is a free parameter and the degree k of the polynomials is numerically equal to $|Q|$.

For $k = 1$ the potential or static energy density is

$$\varepsilon = 2 \frac{|\lambda(a-b)|^2}{\left\{|z-b|^2 + |\lambda|^2 |z-a|^2\right\}^2}, \qquad [20]$$

which possesses a bell-like shape whose maximum value

$$\varepsilon_{\max} = 8 \frac{\left(|\lambda|^2 + |1|\right)^2}{|\lambda(a-b)|^2} \qquad [21]$$

is positioned at

$$z_{\max} = \frac{a|\lambda|^2 + b}{|\lambda|^2 + 1}. \qquad [22]$$

In any given topological sector the static energy is minimised when one of the Cauchy conditions is satisfied. A solution of [18] automatically solves the original second order equation [14] but the converse need not be true. However, all the static finite-energy solutions of [14] are exhausted by equation [18] (6, 7). This is a special asset of the CP^1 model on S_2 which is absent in its generalisation CP^n. The latter possesses static solutions like $W(z, \bar{z})$ which are non-meromorphic and correspond to saddle points of the energy (8). Furthermore, the CP^1 model itself on a torus has solutions to [14] which disobey [18]. For the model a torus see (9).

Viewed as an evolving structure the soliton [19] is unstable under any small perturbation, either explicit (e.g., by setting the soliton into motion) or implicit (as inevitably introduced by the discretisation procedure). For fields $W = \lambda(z-a), \lambda(z-a)(z-b)$, etc, simpler in form than [19], such behaviour has been seen both in the full simulation of the model and in the collective coordinate approximation (10, 11). The said instability, which eventually collapses the numerical procedure by infinitely shrinking or expanding the soliton, is associated with the conformal invariance of the *O(3)* lagrangian in two dimensions: the solitons can change their size at the expense of no energy at all.

As commented earlier on, the instability of the discretised *O(3)* model can be cured by the addition of two extra terms to the lagragian [2]. The first one resembles the term introduced by Skyrme in his nuclear model in four dimensional space-time, and the second one is a potential-like term. Such modified 'baby Skyrme model' possesses stable lumps which also scatter at π / N.

Modified model

Our modified model corresponds to the lagrangian density

$$\mathcal{L}_s = \mathcal{L}_o - \tfrac{1}{4}\theta_1\left[\left(\partial^\mu\vec{\phi}\cdot\partial_\mu\vec{\phi}\right) - \left(\partial^\mu\vec{\phi}\cdot\partial^\nu\vec{\phi}\right)\left(\partial_\mu\vec{\phi}\cdot\partial_\nu\vec{\phi}\right)\right]$$
$$-\tfrac{1}{4}\theta_2\left[-2\mathrm{Re}(\lambda)\phi_1 - 2\mathrm{Im}(\lambda)\phi_2 + \left(1-|\lambda|^2\right)\phi_3 + \left(1+|\lambda|\right)^2\right]^4$$

where $\mathcal{L}_0$ is given by equation [2] and $\theta_1, \theta_2 \in \Re^+$. The θ-terms break the conformal invariance and their combined effect stabilises the solitons. If the size of the solitons is appropriately chosen, it is energetically unfavourable for them to change it. Also note that the above Lagrangian is no longer *O(3)* invariant, but it respects the requirement of relativistic invariance. Both the kinetic and potential energies can be easily read-off from [23]. The latter rescales as

$$V_s = \left[\vec{\phi}(\gamma\vec{x})\right] = V_0\left[\vec{\phi}(\vec{x})\right] + \gamma^2 V_{\theta_1}\left[\vec{\phi}(\vec{x})\right] + \gamma^{-2} V_{\theta_2}\left[\vec{\phi}(\vec{x})\right], \quad [24]$$

in obvious notation: V_0 is the scale free potential [6]; V_{θ_1} is the Skyrme-like term that prevents the solitons from shrinking whereas the V_{θ_2} term resembles a potential that prevents their expansion. Judicious choices of the θ_2 term, which unlike the Skyrme term is nonunique (12), opens up the possibility of writing different interesting versions of the baby skyrmion model, a realisation of which is [23].

Through laborious but straightforward manipulation one can cast [23] into the more tractable W formulation. We get the field equation

$$\begin{aligned}0 &= W_{tt} - W_{xx} - W_{yy} - \frac{2W}{|W|^2+1}\left[(W_t)^2 - (W_x)^2 - (W_y)^2\right] + \\ &\frac{4\theta_1}{\left(|W|^2+1\right)^2}\Big\{2\overline{W}_{tx}W_tW_x + 2\overline{W}_{ty}W_tW_y - 2\overline{W}_{xy}W_xW_y - \overline{W}_{tt}\left[(W_x)^2 + (W_y)^2\right] + \\ &\overline{W}_{xx}\left[(W_y)^2 - (W_t)^2\right] + \overline{W}_{yy}\left[(W_x)^2 - (W_y)^2\right] + W_{tt}\left(|W_t|^2 - |W_y|^2\right) + \\ &W_{xx}\left(|W_t|^2 - |W_y|^2\right) + W_{yy}\left(|W_t|^2 - |W_x|^2\right) - W_{tx}\left(\overline{W}_tW_x - W_t\overline{W}_x\right) - \\ &W_{ty}\left(\overline{W}_tW_y + W_t\overline{W}_y\right) + W_{xy}\left(\overline{W}_xW_y + \overline{W}_yW_x\right) + \\ &\frac{2W}{|W^2|+1}\left[\left(\overline{W}_tW_x - W_t\overline{W}_x\right)^2 + \left(\overline{W}_tW_y - W_t\overline{W}_y\right)^2\right] - \Big\} \\ &\frac{4\theta_2|W-\lambda|^6}{\left(|W|^2+1\right)^2} \times \left[\bar{\lambda}W^2 + \left(1-|\lambda|^2\right)W - \lambda\right]\end{aligned} \quad [25]$$

For the static case we drop all time-dependent terms in [25] and, also ignoring the anti-soliton terms like $W_{\bar{z}}$, the above equation simplifies to

$$\begin{aligned}0 = 8\Big\{&\theta_1|W_z|^4 - 4\theta_2|v|^2 \\ &-\left[4\theta_1(W_z)^2\overline{W}_{zz} - \theta_2 v\frac{d\bar{v}}{d\overline{W}}\overline{W}\right]\Big\}W \\ &-\left[4\theta_1(W_z)^2\overline{W}_{zz} - \theta_2 v\frac{d\bar{v}}{d\overline{W}}\right]\end{aligned} \quad [26]$$

where $v \equiv (W-\lambda)^4$. It is directly checked that the configuration

$$W = \lambda\frac{z-a}{z-b} \quad [27]$$

where

$$\lambda = \frac{\sqrt[4]{2\theta_1/\theta_2}}{a-b}, a, b, \in C, \quad [28]$$

solves the equation of motion. The field [27] is the familiar expression for a single CP^1 soliton but now with λ fixed by [28]. A soliton with its size thus fixed is sometimes called a 'baby skyrmion' [an anti-skyrmion would be the complex conjugate of [27]]. It is noteworthy that theories like the tHP monopoles have a parameter similar to λ, which determines the size of the monopoles.

The skyrmion's potential or static energy density E is found by inserting [27] into

$$E = 2\frac{|W_z|^2 + |W_{\bar{z}}|^2}{\left(|W|^2+1\right)} + 8\theta_1\frac{\left(|W_z| - |W_{\bar{z}}|^2\right)^2}{\left(|W^2|+1\right)} + 4\theta_2\frac{|W-\lambda|^8}{\left(|W|^2+1\right)^4}, \quad [29]$$

whose maximum value is

$$E_{\max} = \varepsilon_{\max}\left(1 + \theta_1\varepsilon_{\max}\right) \; \varepsilon_{\max} = 8\frac{\left(|\lambda|^2+1\right)^2}{|\lambda(a-b)|^2} \quad [30]$$

The position $z_{\max}$ is still determined by formula [22]. Observe that in the limit where the θ´s go to zero we recover the pure *O(3)* model.

Putting $v = 1$ in equation [26] gives

$$E_{\max} = 2|\lambda|^2 + 8\theta|\lambda|^4 4\theta_2, \lambda = \sqrt[4]{\frac{\theta_2}{2\theta_1}}$$

which is a peak value located at $z_{\max} = a$. This important particular case was tackled in reference (13) and corresponds to a soliton of the form $W = \lambda(z-a)$.

In order to study processes involving two skyrmions we are going to consider fields of the appearance

$$W = \lambda\frac{z-a}{z-b}\frac{z+c}{z+d}, \quad [31]$$

which do not satisfy identically the field equation [26] and hence describe two solitons only in an approximate manner.

Numerical procedure

We treat the fields W as the basic initial configurations and their analytic values are used at each lattice site in the discrete approximation of the model. After Lorentz boosting W, we pass on to the ϕ-formulation by means of equation [13]. Then $\vec{\phi}$ is numerically evolved according to the full equation [25], written in terms of $\vec{\phi}$.

It is quite common for ϕ_3 to have values near to 1, in which case W becomes too large for numerical comfort. So in our simulations we have preferred, instead of [13], the stereographic projection

$$W' = \frac{1-\phi_3}{\phi_1 + i\phi_2}. \qquad [32]$$

The dynamics of the system is unchanged because $\mathcal{L}(W') = \mathcal{L}(W)$.

Our simulations employ the fourth-order Runge-Kutta method and approximate the spatial derivatives by finite-differences. The laplacian is evaluated using the standard nine-point formula. We use double-precision arithmetics on a 200×200 ($n_x = n_y = 200$) lattice with spatial and time steps $\delta x = \delta y = 0.02$ and $\delta t = 0.005$.

Unavoidable numerical truncation errors introduced at various stages of the calculations gradually shift the fields away from the unit sphere [1], thereby building-up numerical inaccuracies in the evolution equations. So we rescale

$$\vec{\phi} \to \frac{\vec{\phi}}{\sqrt{\vec{\phi}\cdot\vec{\phi}}}$$

every few iterations. The error associated with this procedure is of the order of the accuracy of our calculations. Each time, just before the rescaling operation, we evaluate the quantity

$$\mu \equiv \vec{\phi}\cdot\vec{\phi} - 1$$

at each lattice point. Treating the maximum of the absolute value of μ as a measure of the numerical error, we found that $|\mu|_{\max} \approx 10^{-8}$. This magnitude is useful as a guide to determine how reliable a given numerical result is. Usage of an unsound numerical procedure like, say, taking $\delta x < \delta t$ in the Runge-Kutta evolution, shows itself as a rapid growth of max $|\mu|$; such increase also occurs when the solitons become exceedingly thin.

We include along the boundary a narrow strip to absorb the various radiation waves, reducing their effect on the skyrmions via the reflections from the boundary. The absorption is implemented by setting

$$\partial_t \vec{\phi} \to \chi \partial_\tau \vec{\phi},$$

where the damping function χ has the form

$$\chi(j) = \begin{cases} 1,\ j \in [0, j_1] \\ 1 - \dfrac{j - j_1}{j_2 - j_1} \\ j \in [j_1 + 1, j_2 - 1] \\ \chi_0,\ j \in [j_2, n_\chi] \end{cases} \qquad (\chi_0 = 0.95)$$

where the absorbing band is no more than about 10 % of mesh-points. The damping device is useful when studying soliton stability, but it is dispensable when considering collision processes.

For the parameters we have chosen

$$a = c = 0.75, \quad b = d = 0.05,$$
$$\theta_1 = 0.015006250, \quad \theta_2 = 0.1250 \qquad [33]$$

The global $U(1)$ symmetry of [27] has been used to choose λ real. In our case, from [28] and [33] it follows that $\lambda = 1$.

One skyrmion

Let us consider the single-skyrmion field

$$W = \frac{z - 0.75}{z - 0.05}. \qquad [34]$$

For an initial velocity equal to naught our simulations show that the energy density corresponding to the above soliton evolves only very slighty and does not change its shape. At the initial time the amplitude of the energy density has the numerical value of ≈ 128.47, but it quickly re-adjusts itself and stabilises around the analytical result as calculated from equation [30] and [33]:

$$E_{\max} \approx 65.3 + 31.99 + 31.99 \approx 129.3$$

Some radiation waves are emitted by the soliton-hump as time goes by. They propagate out to the boundary at the speed of light leaving the central region of the lattice essentially free of kinetic energy. The smallness of the kinetic energy density indicates that our soliton is almost perfectly static; this is in fact numerically observed: at the initial time the lump of energy is situated at $z_{\max} = (0.40{,}0)$ and by t=10 it has slowly shifted to (0.4013,0). Note that the theoretical value of $z_{\max}$, as per formula [22], is precisely (0.40,0). The kinetic energy density decreases as time goes by and fades away, due to the absorption set-up operating along a small band near the edges of the grid (see section 5). A cursory glance at [23] shows that by setting $\theta_1 = \theta_2 = 0$ we recover the pure *O(3)* model:

$$\lim_{\theta_1, \theta_2 \to 0} \mathcal{L}_{\text{sky}} \to \mathcal{L}_{O(3)} \qquad [35]$$

Upon effectively moving the boundaries to infinity, our simulations for this limiting case show that [34] represents a static *O(3)* solution which, however, is unstable on the mesh, corroborating the results found in (10).

It is most interesting that the limit [35] resembles the BPS limit in the tHP monopole theory.

Two solitons

We now shift our attention to the two-soliton configuration

$$W = \frac{(z - 0.75)\ (z + 0.75)}{(z - 0.05)\ (z + 0.05)}, \qquad [36]$$

which gives two skyrmion-lumps of equal size initially well separated from each other but still far away from the edges of the mesh, thus avoiding reflections from the boundaries as much as possible. As pointed out at the end of section 4 the state [36] is not an exact solution of the model, therefore it should undergo some evolution even for an initial speed of zero.

The amplitude starts at a value somewhat bigger than twice the value for a single soliton. As soon as the evolution commences the skyrmions shake off some radiation and alter their size by getting broader. In so doing they slowly move away from each other, unveiling the presence of a repulsive force between them.

During this process the peak $E_{\max}$ decreases and undergoes damped oscillations around the canonical value 129.3; by $t \approx 8$ the oscillations are quite small and the energy stabilises near that value. The kind of weak repulsion just described has also been observed using a collective coordinate method (14).

In the limit [35] we have verified that the repulsion between the lumps disappears and they remain motionless in their initial positions throughout the simulation. However, in this case the solitons are unstable and their energy density increases non-stoppingly: their breadth goes down to the order of the lattice spacing, eventually collapsing the numerics.

Next we study head-on collisions between two skyrmions of the form [36]. Upon boosting the solitons towards each other there is always an initial burst of radiation and the lumps re-arrange themselves around their stable size. At small speeds the two humps approach each other but the repulsive force between them results in their motion being reversed.

A qualitatively similar behaviour is observed for speeds up to approximately 0.3. For $v \approx 0.3$ and higher the skyrmions acquire enough kinetic energy to overcome their mutual repulsion; during their collision they form a complicated ringish state (where they attain a minimum height and hence maximum width) and re-emerge at 90° with respect to the original direction of motion in the centre-of-mass frame. The emerging skyrmions are initially shrinking but, after they have travelled some distance, they expand once more. The final state is achieved after some oscillations of the energy density.

The existence of a critical velocity above (below) which the lumps scatter at right angles (backwards) is a major difference between the pure *O(3)* model and its modified Skyrme version: in the limit [35] we have been able to confirm that this critical velocity ceases to exist and 90° scattering occurs as long as $v > 0$. In the pure model events unfold very much like before but now the system in no longer stable. The soliton-humps continue to grow thin, increasing in height, eventually breaking down the numerical procedure.

It is noteworthy that a similar scattering behaviour, and the existence of a critical velocity as well, are exhibited by other important soliton models like vortices (15) and monopoles (16).

For collisions with small but non-zero impact parameter the results are not at variance with prognostication: the skyrmion-lumps scatter either bouncing back or at nearly right-angles to the initial direction of motion, depending on the velocity. In general, the larger the impact parameter the smaller the scattering angle, and the more the lumps conserve their identity during the process.

Conclusions

Restricting ourselves to solitons with topological charge Q= 1,2 we have performed a numerical study of some stability and scattering properties of the CP^1 model in (2+1) dimensions, both in its pure and Skyrme versions.

We have found that the model is stable only in its Skyrme version. The so-called ‘baby skyrmions’ possess energy density profiles that do not change appreciably in shape, nor they shrink or expand unduly with the passing of time.

The single-skyrmion case is almost perfectly static, whereas for $Q = 2$ there is a repulsive force between the lumps. This repulsive interaction is responsible for the skyrmions to scatter back-to-back if the initial, boosting velocity is smaller than a certain critical value. Otherwise they scatter at 90°. Our Skyrme version of the model is a low dimensional analogue of the nuclear skyrmion theory in four dimensional space-time.

For pure *O(3)* solitons, a limiting case of the Skyrme theory, scattering at 90° takes place for any non-zero initial velocity, i.e., the soliton-lumps no longer repel each other. This is confirmed in the static $Q = 2$ case where the bell-shaped quasi-particles keep still as time goes by, before the instability breaks down the numerical code. In the pure scheme the solitons are not stable because they are invariant under scale transformations. Any perturbation causes the solitons’ energy density to increase without limit. When its breadth is comparable to the lattice spacing the numerical code breaks down. An explicit perturbation can be introduced into the system by impinging the solitons with some initial velocity, but the implicit perturbation inevitably brought about by the discretisation procedure suffices to trigger off the instability.

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