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Effective potential for non-coupled stochastic partial differential equations*

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Abstract

We provide an Effective Potential Formulation for non-coupled (i.e. independent) Stochastic Partial Differential Equations with additive noise using the Hojman et al. method for writting the associated Lagrangean function. We show that this Potential is able to reproduce all the dynamics of the system, once a special differential operator has been applied. This procedure can be used to study the equilibria cases, the complete time evolution and spatial inhomogeneities of the system under consideration.

Key words: stochastic partial differential equations, effective potential.

Potencial efectivo para ecuaciones parciales diferenciales estocásticas no-locales

Resumen

Se provee una formulación de potencial efectivo para ecuaciones diferenciales parciales estocásticas no acopladas (i.e. independientes) con ruido aditivo, usando el método de Hojman *et al.*, para escribir la función lagrangiana asociada. Se muestra que este potencial permite reproducir toda la dinámica del sistema. Este procedimiento es usado para estudiar los casos de equilibrio, la evolución completa en el tiempo y las inhomogeneidades espaciales del sistema bajo consideración.

Palabras claves: ecuaciones diferenciales parciales estocásticas, potencial efectivo.

1. Introduction

There exists a wide spectrum of phenomena (1-6) for which self-regulation, oscillation, adaptiveness and multi-equilibrium behaviours can be adequately modeled by means of stochastic partial differential equations (SPDE). Due to its importance and versatility, efforts have been made to find a formalism from which all the relevant physical information of the system can be obtained (7-12).

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It is widely accepted that a variational principle cannot be constructed for an arbitrary differential equation (13). There is a strict mathematical theorem that shows its existence for a given situation, and whose application reduces the number of equations in physics that have a Lagrangean-Hamiltonian formulation (14,15). However, there are several methods in the literature devised to circumvent this condition and even some proposals for modified variational principles (13, 16-18). Another approach, that has its historical origin in the times of Helmholtz (19), consists in studying the existence and uniqueness (or multiplicity) of Lagrangeans for systems of differential equations (20, 21). One important result in this field is that of Hojman et al. (22), who have proven that it is possible to construct the Lagrangean for any regular mechanical system as a linear combination of its own equations of motion. This particular construction is much wider than the traditional definition $L=T-V$, which is only true when the "forces" involved are derivable from position-dependent potentials (or in some cases from velocity-dependent potentials), therefore it may be used for general non-conservative systems. Its application to the study of SPDEs may provide additional understanding of the internal structure of these phenomena and also enables the use of a well known mathematical machinery to find conserved quantities, equilibria and stability cases, and other dynamical properties. To view an example of the application of this method to self-regulated systems, refer to (23).

The aim of this paper is to provide an alternative procedure to find effective potentials associated to SPDEs with arbitrary additive noise function via a variational formalism constructed by means of the Hojman et al. method.

2. Variational Approach

We will consider in this work SPDEs that can be written as

$$\Box\Psi^i - \Phi^i\left(\Psi^j\right) - \xi^i = 0 \qquad [1]$$

for $i,j=1,\ldots,m$ where m is the number of degrees of freedom, $\Psi^i \equiv \Psi^i(q^j,t)$ denotes the components of a vector field whose arguments are, in the general case, spatial coordinates q^j and time t; $\Box$ is an arbitrary linear space or time (or both) differential operator that does not depend on the field Ψ^i; $\Phi^i(\Psi^j)$ is any, ussually non-linear, deterministic forcing term and $\xi^i \equiv \xi^i(q^j,t)$ is a random function of its arguments describing the stochastic force (noise) in the system. Hereafter, Einstein summation convention and Euclidian metric tensor are assumed (table 1).

Table 1
Some Operators and Dissipation Functions.

Operator	$\Box$	Function	$\Phi^i(\Psi^j)$
D'Alembertian	$\partial_t^2 - \nabla^2$	Polynomial	$\alpha_p[\Psi^i]^p$
Diffusion	$\partial_t - k\nabla^2$	Burgers (noisy)	$\frac{\omega}{2}(\nabla\Psi^i)^2$
Temporal	∂_t	Purely Dissipative	$-\frac{\delta H(\Psi^j)}{\delta\Psi^i}$

The equation of motion for a mechanical system arises from a set of m differential equations. Whereas the equations [1] can be understood as equations of motion in a variational sense, it is possible to write

$$G^i \equiv \ddot{\Psi}^i - F^i(\dot{\Psi}^j, \partial_\kappa, \Psi^j, \Psi^j, t) = 0 \qquad [2]$$

where F^i behaves as "forces" (both deterministic and stochastic) divided by unitary mass and may include spatial derivatives of the field; the dot means total temporal derivative.

In the Inverse Problem of the Variational Calculus the Lagrangean $L(\dot{\Psi}^i, \Psi^i, t)$ is constructed such that relations [2] can be effectively deduced via Euler-Lagrange equations. The existence of such a Lagrangean is studied using the Helmholtz conditions.

$$\left\{ \begin{array}{c} \dfrac{\partial G^i}{\partial \ddot{\Psi}^j} = \dfrac{\partial G^j}{\partial \ddot{\Psi}^i} \\ \dfrac{\partial G^i}{\partial \dot{\Psi}^j} = \dfrac{\partial G^j}{\partial \dot{\Psi}^i} = \dfrac{d}{dt}\left(\dfrac{\partial G^i}{\partial \Psi^j} + \dfrac{\partial G^j}{\partial \dot{\Psi}^i} \right) \\ \dfrac{\partial G^i}{\partial \Psi^j} - \dfrac{\partial G^j}{\partial \Psi^i} = \dfrac{1}{2}\dfrac{d}{dt}\left(\dfrac{\partial G^i}{\partial \dot{\Psi}^j} = \dfrac{\partial G^j}{\partial \dot{\Psi}^i} \right) \end{array} \right\} \quad [3]$$

Nevertheless, these conditions do not give any warranty about uniqueness. Two Lagrangeans are said to be solution-equivalent (or s-equivalent) if they differ only by a global multiplicative constant, η, and a total time derivative of some gauge $\Lambda\left(\dot{\Psi}^i, \Psi^i, t\right)$:

$$\eta L = \tilde{L} - \frac{d\Lambda}{dt} \quad [4]$$

The different systems of equations they provide, however, have exactly the same equations of motion.

The Hojman et al. method enables us to write L as a linear combination of the known equations of motion; then for $i,j=1,\ldots,m$,

$$\tilde{L} = m_i\left[\ddot{\Psi}^i - F^i\right] \quad [5]$$

where

$$\mu^i(\dot{\Psi}^i, \Psi^i, t) \equiv D_1 \frac{\partial D_2}{\partial \dot{\Psi}^i} + \ldots\ldots + D_{2m-1}\frac{\partial D_{2m}}{\partial \dot{\Psi}^i} = -\frac{\partial \Lambda}{\partial \dot{\Psi}^i} \quad [6]$$

In equation [6] the quantities under partial derivative (D_{2m}) are constants of motion of the system, while the corresponding coefficients (D_{2m-1}) are arbitrary functions whose arguments are constants of motion. There are plenty of ways to write these (D_{2m}) functions. For instance, one possible form for the (D_{2m-1}) functions, given the (D_{2m}) conserved quantities, is presented in reference (20).

It is important to remark that this method is useful for both second-order and first-order differential equations (22). For further details the reader is exhorted to review (20, 22) and the references therein.

By virtue of equation [5], the general Lagrangean for SPDEs of the form [1] can be written as

$$L = \mu^i(\Box\Psi^i - \Phi^i - \xi^i) \quad [8]$$

where the μ^i parameters must be determined for each case in study.

Once the Lagrangian [8] is completely determined, the corresponding Hamiltonian can be found trivially by the usual Legendre transformation, and, as will be shown in the following lines, also the effective potential can be written straightforwardly.

3. The Effective Potential

For the sake of simplicity, in what follows we will consider the cases for which the derivatives $\frac{d\Psi^i}{d\Psi^j}$ are always zero.

Let us define now the following differential operator

$${}^*\nabla_i \equiv \frac{1}{\dot{\Psi}^j}\left(\frac{d}{dt}\right)_i^j \quad [9]$$

Then, let require that the effective potential, V_{eff}, be such that

$${}^*\nabla_i V_{eff} = -F_i \quad [10]$$

Note that if $V_{eff} = V_{eff}(\dot{\Psi}^i, \Psi^i, t)$, then

$$\left(\frac{d}{dt}\right)_i^j \equiv \delta_i^j \frac{\partial}{\partial t} + \dot{\Psi}^j \frac{\partial}{\partial \Psi^i} + F^j \frac{\partial}{\partial \dot{\Psi}^i} \quad [11]$$

and thus, in the conservative case, i.e. $V_{eff} \cong V_{eff}(\Psi) \cong V$, equation [10] provides (from now on the symbol $\cong$ indicates that the relation is valid only in special cases)

$${}^*\nabla_i V_{eff} \cong \frac{\partial V}{\partial \psi^i} = -F_i \quad [12]$$

which is equivalent to the standard relation $\nabla_i V = -F_i$ when the vector field Ψ^i coincides with the spatial coordinate q^i.

Now, we are interested in finding a gauge $\Lambda = \lambda$ such that equations [8] and [5] provide an s-equivalent Lagrangean, *L*, that can be written as the difference between some function (kinetic energy) T_{eff} and the effective potential V_{eff}. In consequence (for the sake of simplicity, $\eta = 1$)

$$V_{eff} = T_{eff} - \mu_i(\Box\Psi^i - \Phi^i - \xi^i) + \frac{d\lambda}{dt} \qquad [13]$$

For a general non-conservative system, T_{eff} can be written as (25)

$$T_{eff} = \alpha A(\Psi^i) + \beta B(\dot{\Psi}^i, \Psi^i) + \gamma C(\dot{\Psi}^i\dot{\Psi}_i, \Psi^i) \qquad [14]$$

and thus, it is necessary to determine the scalars α, β, γ and the functions $A(\Psi^i), B(\dot{\Psi}^i, \Psi^i), G(\dot{\Psi}^i\dot{\Psi}_i, \Psi^i)$ to completely define V_{eff}.

It should be useful to write the kinetic energy of the system in the traditional quadratic form

$$T_{eff} \cong \frac{1}{2}\dot{\psi}^1\dot{\psi}_1 \qquad [15]$$

In order to do so, it is necessary to find the corresponding gauge first. Equation [10] provides the necessary constraint; thus, by taking the nabla-star derivative at both sides of equation [13] we obtain, for non-coupled equations,

$$*\nabla\left[\frac{d\lambda}{dt} - \mu_j\left(\Box\Psi^i - \Phi^i - \xi^i\right)\right] = -\left(F_i + \ddot{\Psi}_i\right) \qquad [16]$$

Solving this equation for the time derivative of the gauge λ, we have

$$\frac{d\lambda}{dt} = -\int \Psi^i(F_i + \ddot{\Psi}_i)dt + \mu_i(\Box\Psi^i - \Phi^i - \xi^i) + V_0 \qquad [17]$$

where V_0 is an arbitrary constant of integration. Consequently, following equation [13]

$$V_{eff} = V_0 + \frac{1}{2}\dot{\Psi}^i\dot{\Psi}_i - \int \Psi^i(F_i + \ddot{\Psi}_i)dt \qquad [18]$$

which is the general effective potential for equation [1] given the choice [15] for the kinetic energy.

4. Concluding Remarks

In summary, we have presented in this work a novel way to provide both variational and effective potential formulations for general SPDEs with *arbitrary* additive noise function. There are several useful applications of this result. Once the Hamiltonian is obtained, for example, quantization of systems described by equation [1] follows straightforwardly. Also, the Hamilton-Jacobi approach may help to solve the equations of motion of a system via a convenient, if possible, variable separation.

The effective potential [13] is constructed such that it contains all the dynamical information of the system. In general, it may have explicit dependence on time, on the field itself and on the derivatives of the field; certainly, the nabla-star operator identifies the contribution of each functional dependence by means of specific terms, as can be seen in equation [11].

As reported (11,12) in other approaches the effective potential is useful only for stationary or static regimes; the approach presented here can be used to study those cases, the complete temporal evolution of [1] and also the equilibria and stability states. The present potential formulation is based on a classical mechanics approach, there is no need of auxiliary or ghost fields. Nor Fokker-Planck equations neither special assumptions about the additive noise function are invoked in the construction.

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