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Approximate solutions of difference systems with advanced arguments

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Abstract

In this paper we give a method for computing the solutions of the equations with advance $y(n+1) = A(n)y(n) + B(n)y(n+2)$, belonging to a specific sequential space.

Key words: Approximate solutions; difference equations with advance.

Soluciones aproximadas de sistemas diferenciales con argumentos avanzados

Resumen

En este artículo suministramos un método computacional para las soluciones de las ecuaciones con avance $y(n+1) = A(n)y(n) + B(n)y(n+2)$, que pertenecen a una secuencia espacial especifica.

Palabras clave: Ecuaciones diferenciales con avanzada; soluciones aproximadas.

1. Introduction

Recently, differential equations with advanced arguments have attracted the attention of applied mathematicians by their possible applications to industrial problems (1, 2) and engineering design of automatic mechanisms.

From the theoretical point of view, these equations present serious obstacles of research, among them, perhaps the most important, those concerning the existence and uniqueness of the solution of the initial value problem (3). Essentially this is due to the absence of a theory of differential inequalities with advanced arguments. Nevertheless, there exists an increasing amount of published papers in this field, a fact that reveals the interest in these equations. This paper continues the study accomplished in (4-9), where the research of these equations is accomplished in specific sequential spaces that are apriori determined. Concretely, in (4) the authors study the difference equations

$$y(n+1) = A(n)y(n) + B(n)y(g(n)), \qquad [1]$$

where the sequence of advanced times $\{g(n)\}$ satisfies $g(n) \geq n+1$. In this paper we will limit our attention to the particular case $g(n) = n+2$.

We will refer to three theorems of existence and uniqueness proven in (4). Previously, we introduce some notations: In what follows **V** will denote the scalar field **R** or **C**. $|x|$ will be used to denote any convenient vector norm in $\mathbf{V}^r$; for an r x r matrix A, $|A|$ will

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denote the corresponding matrix norm. the following sequential spaces are used in our paper

$$\ell\infty = \{f : N \to V^r : |f|^\infty < \infty\},\ |f|^\infty = \sup\{|f(n)| ; n \in \mathrm{N}\},$$

and

$$\ell^1 = \{f : N \to V^r, |f|^1 < \infty\},\ |f|^1 = \sum_{n=0}^{\infty} |f(n)|.$$

For a sequence $\{F(n)\}$ of invertible matrices we will define

$$\ell^\infty F = \{f : N \to V^r : F^{-1} f \in \ell^\infty\},\ |f|_F^\infty = |F^{-1} f|^\infty$$

$$\ell^\infty F = \{f : N \to V^r : F^{-1} f \in \ell^1\},\ |f|_F^1 = |F^{-1} f|^1.$$

Finally, for integers $m \le n$, we will denote $\overline{m, n} = \{k : m \le k \le n\}$.

Let Φ denote the fundamental matrix of the equation

$$x(n+1) = A(n)x(n), \qquad [2]$$

where $\{A(n)\}$ is a sequence of r x r invertible matrices.

Theorem A. *If*

(C1) $$\sum_{m=0}^{n} |\Phi(n)\Phi^{-1}(m)| \le M, \forall n \in \mathrm{N}$$

and $M |\{B(n)\}|^\infty < 1$ *hold, then for each initial condition* $\xi \in V^r$ *the initial condition problem*

$$\begin{cases} y(n+1) = A(n)y(n) + B(n)y(n+2) \\ y(0) = \xi \end{cases} \qquad [3]$$

has a unique solution in the sequential space ℓ^∞.

Theorem B. *If*

(C2) $$|\Phi(n)\Phi^{-1}(m)| \le K\alpha(n)\alpha(m)^{-1},\ n \ge m,$$

where $\{\alpha(n)\}$ *is a sequence of positive numbers, and*

$$K \sum_{m=0}^{\infty} \alpha(m+2)\alpha(m+1)^{-1} |B(m)| < 1 \qquad [4]$$

hold, then for each initial condition $\xi \varepsilon V^r$ *the initial condition problem* [3] *has a unique solution in the sequential space* ℓ_α^∞.

Theorem C. *If*

(C3) $$\sum_{m=0}^{\infty} |\Phi^{-1}(m+1)B(m)\Phi(m+2)| < 1$$

holds, then for each initial condition $\xi \varepsilon V^r$ *the initial value problem* (3) *has a unique solution in the sequential space* ℓ_Φ^∞.

Each of these theorems imply that , the generating matrix of solutions of Equation [1], exists in the respective sequential space and a method of an asymptotic decomposition of can be developed (4). Nevertheless, the asymptotic of obtained seems to be more useful in theoretical questions (4, 5) rather than in concrete approximations of solutions of Equation (1).

In this paper we are concerned with the approximate computation of the solutions of [3] given by the above theorems. We will show that we can calculate the solutions of [3] by fixed points of finite dimensional operators.

2. Approximation of solutions in ℓ_α^∞

In this section we will solve the following problem: Given ε, a positive number, find an interval of integers $\overline{o, N}$ and a function $\omega_N : \overline{o, N} \to V^r$ such that $\omega_N(0) = \xi$ and

$$|\omega(n, \xi) - \omega_N(n)| < \varepsilon,\quad \forall n \in \overline{0, \mathrm{N}},$$

where $\omega(n, \xi)$ is the solution of [3] in the space ℓ^∞. We will abbreviate $E(m) = \Phi^{-1}(m+1)B(m)$.

Let us consider the operator

$$T_N : S(N,V) \to S(N,V),$$

where $$S(N,V) = \{\omega : \overline{0, N} \to V^r,$$

and T_N is defined by

$$T_N(\omega)(n) = \Phi(n)\xi + \sum_{m=0}^{n-1} \Phi(n)E(m)\omega(m+2),$$
$$0 \le n \le N-1$$

and

$$T_N(\omega)(n) = \Phi(n)\xi + \sum_{m=0}^{n-2} \Phi(N)E(m)\omega(m+2) +$$
$$B(N-1)\omega(1).$$

If we endow the finite dimensional space $S(N, V)$ with the norm

$$|\omega|_N = \max\{|\omega(j)|; 0 \le j \le N\},$$

then from **(C1)** we obtain the estimate

$$|T_N(\omega_1) - T_N(\omega_2)|_N \le M\,|\{B(n)\}|^\infty |\omega_1 - \omega_2|_N .g4$$

If $M\,|\{B(n)\}|^\infty < 1$, then the operator T_N has a unique fixed point in $S(N, V)$ we will denote by ω_N. Furthermore by using the condition **(C1)** we estimate $|\omega_N|_N$.

Therefore

$$|\omega_N|_N \le \frac{M\,|\xi|}{1 - M\,|\{B(n)\}|^\infty}. \qquad [5]$$

We recall from (4) that under condition **(C1)**, the solution is the fixed point of the operator

$$T(\omega)(n) = \Phi(n)\xi + \sum_{m=0}^{n-1} \Phi(n)E(m)\omega(m+2)$$

in the space ℓ^∞. From **(C1)** and $|\{B(n)\}|^\infty < 1$ we obtain the following estimate

$$|\hat{\omega}(.,\xi)|^\infty \le \frac{M\,|\xi|}{1 - M\,|\{B(n)\}|^\infty}. \qquad [6]$$

In what follows we will assume that is imbedded in ℓ^∞: the sequence $\omega \in S(N, V)$ is identified with $\hat{\omega} \in \ell^\infty$ defined by

$$\hat{\omega}(n) = \begin{cases} \omega(n), & 0 \le n \le N \\ 0, & n < N \end{cases}.$$

According to definitions of T and T_N, if $w\omega \in S(N, V)$, them

$$T(\hat{\omega})(n) - T_N(\omega)(n) = 0, \ if\ 1 \le n \le N-1,$$

and

$$T(\hat{\omega})(n) - T_N(\omega)(N) = -B(N-1)\omega(1).$$

From here we obtain

$$\max_{0 \le n \le N} |T(\hat{\omega})(n) - T_N(\omega)(n)| \le |B(N-1)||\omega|_N.$$

Applying this estimate to and using [5] we obtain:

$$\max_{0 \le n \le N} |T(\hat{\omega}_N)(n) - \omega_N(n)| \le \frac{M\,|\xi|}{1 - M\,|\{B(n)\}|^\infty} |B(N-1)|. \qquad [7]$$

Let us consider the problem (3) in the case:

$$\lim_{n\to\infty} B(n) = 0. \qquad [8]$$

Then

$$\max_{0 \le n \le N} |\omega(n,\xi) - \omega_N(n)| \le$$
$$\max_{0 \le n \le N} |T(\hat{\omega}_N)(n) - \omega_N(n)|$$
$$+ \max_{0 \le n \le N} |T(\omega(.,\xi))(n) - T(\hat{\omega}_N)(n)|$$
$$\le \frac{M\,|\xi|}{1 - M\,|\{B(n)\}|^\infty} |B(N-1)|.$$
$$+ M\,|\{B(n)\}|^\infty \max_{0 \le n \le N} |\omega(n,\xi) - \omega_N(n)|$$
$$+ |B(N-1)|\Big[|\omega(.,\xi)|^\infty + |\omega_N|_N\Big]$$

Using [6] obtain

$$\max_{0 \le n \le N} |\omega(n,\xi) - \omega_N(n)| \le \frac{M\,|\xi|}{1 - M\,|\{B(n)\}|^\infty} |B(N-1)|$$

$$+ M\,|\{B(n)\}|^\infty \max_{0 \le n \le N} |\omega(n,\xi) - \omega_N(n)|.$$

$$+ \frac{2M|\xi|}{1 - MM\,|\{B(n)\}|^\infty} |B(N-1)|$$

This inequality implies

$$\max_{0 \le n \le N} |\omega(n,\xi) - \omega_N(n)| \le \frac{3M|\xi|}{(1 - M|\{B(n)\}|^{\infty})^2} |B(N-1)|$$

This estimate proves the following

Theorem 1. *If conditions* **(C1),** [8] *and* $M|\{B(n)\}|^{\infty} < 1$ *are satisfied, then for a given positive number* ε*, there exists an* $N = N(\varepsilon) \in N$ *such that for every* $n \in \overline{0, N}$ *we have* $|\omega(n,\xi) - \omega_N(n)| \le \varepsilon.$

According to the definition of the operator T_N, the equation $\omega = T_N(\omega)$ is equivalent to the following nonhomogeneous linear system:

$$\begin{cases} \omega(1) &= A(0)\xi + B(0)\omega[2]o\kappa \\ \omega(2) &= A(1)\xi + B(1)\omega[3]o\kappa \\ \cdots\cdots & = \cdots\cdots\cdots\cdots\cdots\cdots \\ \omega(N-1) &= A(N-2)\omega(N-2) + B(N-2)\omega(N) \\ \omega(N) &= A(N-1)\omega(N-1) + B(N-1)\omega(1) \end{cases}.$$

This system can be solved, approximately, by the iterative scheme: $x_0 = 0,\ x_k = T_N(x_{k-1})$ converging with speed $(M|\{B(n)\}|^{\infty})^k$, that is

$$\max_{0 \le n \le N} |x_k(n) - \omega_N(n)| \le \left(M|\{B(n)\}|^{\infty}\right)^k |\omega_N|_N$$
$$\le \frac{|\xi|}{1 - M|\{B(n)\}|^{\infty}} \left(|\{B(n)\}|^{\infty}\right)^k.$$

3. Approximation of solution in ℓ_a^{∞}

In this section, we will endow the space $S(N,V)$ with the norm

$$|\omega|_{N,\alpha} = \max\left\{|\alpha(n)^{-1}\omega(n)| : 0 \le n \le N\right.. $$

We will consider $\omega(n,\xi)$, the solution of problem [3] given by Theorem B. We recall that under condition **(C2),** this solution is a fixed point of operator T acting from ℓ_a^{∞} into itself. In order to use condition **(C2),** we will impose the condition

$$\delta = K \sum_{m=0}^{\infty} \alpha(m+2)\alpha(m+1)^{-1}|B(m)| < 1. \qquad [9]$$

Let us contemplate the operator $\mathcal{L}_N : S(N,V) \to S(N,V)$ defined by

$$\mathcal{L}_N(\omega)(n) = (n) = \Phi(n)\xi + \sum_{m=0}^{n-1} \Phi(n)E(m)\omega(m+2),$$
$$0 \le n \le N-1,$$

and

$$\mathcal{L}_N(\omega)(N) = \Phi(N)\xi + \sum_{M=0}^{N-2} \Phi(N)E(m)\omega(m+2)$$
$$+ \frac{\alpha(N)}{\alpha(1)} \Phi(N)\Phi^{-1}(N-1)B(N-2)\omega(1).$$

We have the following property of $\mathcal{L}_N$:

$$|\mathcal{L}_N(\omega_1) - \mathcal{L}_N(\omega_2)|_{N,\alpha}.$$

Since $\delta < 1$, then the operator $\mathcal{L}_N$ has a unique fixed point $S(N,V)$ in that we will denote by ω_N. Moreover, we point out the following estimate

$$|\omega_N|_{N,\alpha} \le \frac{K|\xi|}{1-\delta}. \qquad [10]$$

We recall the following estimate for $\omega(n,\xi)$, the fixed point of operator T in the space ℓ_α^{∞}:

$$|\omega(.,\xi)|_{\alpha}^{\infty} \ge \frac{K|\xi|}{1-\delta}.$$

According to the definitions of T and $\mathcal{L}_N$, for $\omega \in S(N,V)$ we have the property

$$T(\hat{\omega})(n) - \mathcal{L}_N(\omega)(n) = 0, \textit{ if } 0 \le n \le N-1$$

and

$$T(\hat{\omega})(n) - \mathcal{L}_N(\omega)N = \frac{\alpha(N)}{\alpha(1)} \Phi(N)\Phi^{-1}(N-1)B(N-2)\omega(1).$$

From here we obtain $\omega \in S(N.V)$ for

$$\max_{0\le n\le N}\left|\alpha(n)^{-1}(\mathrm{T}(\hat{\omega})(n)-\mathcal{L}_N(\omega)(n))\right| \le \frac{K\alpha(N)}{\alpha(N-1)}\left|B(N-2)\right|\left|\omega\right|_{N,\alpha}.$$

From this estimate and [10] we may write

$$\max_{0\le n\le N}\left|\alpha(n)^{-1}\left(\mathrm{T}(\hat{\omega}_{\mathrm{N}})-\omega_{\mathrm{N}}(n)\right)\right|\le \frac{K^2|\xi|\alpha(N)}{(1-\delta)\alpha(\mathrm{N}-1))}\left|B(N-2)\right|. \quad [11]$$

From [10] and [11] we obtain

$$\begin{aligned}&\max_{0\le n\le N}\left|\alpha(n)^{-1}(\omega(n,\xi)-\omega_{\mathrm{N}}(n))\right|\\ &\max_{0\le n\le N}\left|\alpha(n)^{-1}(\mathrm{T}(\omega(.,\xi))-\mathrm{T}(\hat{\omega}_N))(n)\right|\\ &\le \frac{K^2|\xi|}{1-\delta}\frac{\alpha(N)}{\alpha(N-1)}\left|B(N-2)\right|\\ &+\delta\max_{0\le n\le N}\left|\alpha(n)^{-1}(\omega(n,\xi)-\omega_N(n))\right|\\ &+K\frac{\alpha(\mathrm{N}+1)}{\alpha(\mathrm{N})}\left|B(N-1)\right|\left|\omega\right|_{\alpha}^{\infty}\\ &+\frac{\alpha(\mathrm{N})}{\alpha(\mathrm{N}+1)}\left|B(N-2)\right|\left|\omega_{\mathrm{N}}\right|_{N,\alpha} \quad [12].\end{aligned}$$

From [10] we obtain

$$\max_{0\le n\le N}\left|\alpha(n)^{-1}(\omega(n,\xi)-\omega_{\mathrm{N}}(n))\right| \le \frac{2K^2|\xi|}{(1-\delta)^2}\sum_{m=N-2}^{N-1}\frac{\alpha(m+2)}{\alpha(m+1)}\left|B(m).\right| .$$

This last estimate and condition [9] imply, for a positive ε, the existence of an $N\in\mathrm{N}$ such that

$$\max_{0\le n\le N}\left|\alpha(n)^{-1}(\omega(n.,\xi)-\omega_N(n))\right|\le\varepsilon.$$

This estimate proves the following

Theorem 2. *If conditions* **(C2)**, [8] $\delta<1$ *and are satisfied, then for a given positive number , there exists an* $N=N(\varepsilon)\in\mathrm{N}$ *such that* $\left|\alpha(n)^{-1}(\omega(n.,\xi)-\omega_{\mathrm{N}}(n))\right|\le\varepsilon$ *for every* $n\in\overline{0,N}$.

Now, the equation $\omega=\mathcal{L}_n(\omega)$ is equivalent to the algebraic system of equations

$$\begin{cases}\omega(1) = A(0)\xi+B(0)\omega(2)\\ \omega(2) = A(1)\xi+B(1)\omega(3)\\ \ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \omega(\mathrm{N}-1)=\mathrm{A}(\mathrm{N}-2)\omega(N-2)+\mathrm{B}(N-2)\omega(\mathrm{N})\\ \omega(\mathrm{N}) = \mathrm{A}(\mathrm{N}-1)\omega(N-1)+\dfrac{\alpha(N)}{\alpha(1)}\Phi(N)E(N-2)\omega(1)\end{cases}.$$

This system can be solved by the iterative sequence $x_0=0, x_k=\mathcal{L}_N(x_k-1)$. The speed of the convergence of this scheme is given by the estimate

$$\max_{0\le n\le N}\left|\alpha(n)^{-1}\left(x_k(n)-\omega_N(n)\right)\right|\le\delta^k\left|\omega_N\right|_{N,\alpha}\le\frac{|\xi|}{1-\delta}\delta^k.$$

4. Approximation of solutions in ℓ_{Φ}^{∞}

Let us denote

$$\gamma=\sum_{m=0}^{\infty}\left|\Phi^{-1}(m+1)B(m)\Phi(m+2)\right|.$$

We recall from(4) that under condition **(C3)**, the unique solution $y(n.,\xi)$ of problem [3] in the space ℓ_{Φ}^{∞} can be written in the form

$$y(n.,\xi)=\Phi(n)\omega(n.,\xi), \quad [13]$$

where $\omega(n.,\xi)$ is a fixed point of the operator

$$\mathcal{M}=(\omega)(n)=\xi+\sum_{m=0}^{n-1}(m+1)B(m)\Phi(m+2)\omega(m+2).$$

For a fixed $N\in\mathrm{N}$, let us consider the operator

$$\mathcal{M}_{\mathrm{N}}:S(N,V)\to S(N,V)$$

defined by

$$\mathcal{M}=(\omega)(n)=\xi+\sum_{m=0}^{n-1}(m+1)B(m)\Phi(m+2)\omega(m+2),$$

$$0\le n\le N-1$$

and

$$\mathcal{M}_N = (\omega)(N) = \mathcal{M}_N(\omega)(N-1) + \Phi^{-1}.$$
$$(N)B(N-1)\Phi(N+1)\omega(1)$$

From

$$|\mathcal{M}_N(\omega_1) - \mathcal{M}_N(\omega_2)|_N \le \gamma|\omega_1 - \omega_2|_N,$$

and the condition $\gamma < 1$, we obtain that the operator $\mathcal{M}_N$ has a unique fixed point in $S(N, V)$ we will denote by . For the norm $|\omega_N|_N$ we write the following estimate:

$$|\omega_N|_N \le \frac{|\xi|}{1-\gamma}. \qquad [14]$$

From the definition of operator $\mathcal{M}$ y $\mathcal{M}_N$ we have

$$\mathcal{M}(\hat{\omega})(n) - \mathcal{M}_N(\omega)(n) = 0, if\ 0 \le n \le N-1,$$

and

$$\mathcal{M}(\hat{\omega})(N) - \mathcal{M}_N(w)(N) = \Phi^{-1}(N)B(N-1)\Phi(N+1)\omega(1).$$

From here we obtain

$$\max_{0\le n\le N}|\mathcal{M}(\hat{\omega})(n) - \mathcal{M}_N(\omega)(n)|_N \le |\Phi^{-1}(N)B(N-1)\Phi(N+1)||\omega|_N.$$

Applying this estimate to ω_N and using [14] we may write:

$$\max_{0\le n\le N}|\mathcal{M}(\hat{\omega})(n) - \omega_N| \le \frac{|\xi|}{1-\gamma}|\Phi^{-1}(N)B(N-1)\Phi(N+1)|.$$

In a similar form as [12] was obained, we have

$$\max_{0\le n\le N}|\omega(n., \xi)(n) - \omega_N(n)| \le \frac{3|\xi|}{(1-\gamma)^2}|\Phi^{-1}(N)B(N-1)\Phi(N+1)| \qquad [15]$$

from the convergence of the series given condition **(C3)** we conclude from [15] the following

Theorem 3. *If condition* **(C3)** *is satisfied, then for a given positive number* ε, *there exists an* $N = N(\varepsilon) \in \mathrm{N}$ *such that*

$$\max_{0\le n\le N}|\omega(n, \xi) - \omega_N(n)| \le \varepsilon$$

for every $n \in \overline{0, N}$. *In addition, if we assume condition*

$$|\Phi(n)| \le K, n \ge 0,$$

there we obtain

$$\max_{0\le n\le N}|y(n, \xi) - \Phi(n)\omega_N(n)| \le K\varepsilon.$$

The equation is equivalent to the system of equations

$$\begin{cases} \omega(1) &= \xi + \Phi^{-1}(1)B(0)\Phi(2)\omega(2) \\ \omega(2) &= \omega(1) + \Phi^{-1}(2)B(1)\Phi(3)\omega(3) \\ \ldots\ldots &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \omega(N-1) &= \omega(N-2) + \Phi^{-1}(N-1)B(N-2)\Phi(N)\omega(N) \\ \omega(N) &= \omega(N-1) + \Phi^{-1}(N)B(N-1)\Phi(N+1)\omega(1). \end{cases}$$

This algebraic system can be approximately solved by the iterative scheme: $x_{0=0,}\ x_k = \mathcal{M}_N(x_{k-1})$ converging with a speed given by the estimate:

$$\max_{0\le n\le N}|x_k(n) - \omega_N(n)| \le \gamma^k|\omega_N|_N \le \frac{|\xi|}{1-\gamma}\gamma^k.$$

5. More general systems

The previous method can be applied to the more general equation with advanced argument [1]. The following theorem was proved in (4).

Theorem D. *If condition* **(C1)**, $g(n) \ge n+1$ *and* $M|\{B(n)\}|^\infty < 1$ *are satisfied, then for every* $\xi \in V^r$ *the initial value problem*

$$\begin{cases} y(n+1) = A(n)y(n) + B(n)y(g(n)) \\ y(0) = \xi \end{cases}, \qquad [16]$$

has a unique solution in the space ℓ^∞.

In order to approximate this solution, let us consider the operator

$$\mathrm{T}_{N,g}: S(N, V) \to S(N, V),$$

defined by

$$T_{N,g}(n) = \Phi(n)\xi + \sum_{m=0}^{n-1} \Phi(n)E(m)\omega(g(m)),\ 0 \le n \le N-1,$$

and

$$T_{N,g}(\omega)(N) = \Phi(n)\xi + \sum_{m=0}^{N-2} \Phi(N-1)E(m)\omega(g(m)) + \sum_{j=0}^{g(1)-2} \Phi(N+g(1)-1)\Phi^{-1}(N'+j-1)\omega(j),$$

Under conditions of Theorem D, the operator $T_{N,g}$ is a contraction with a unique fixed point in $S(N,V)$. If condition [8] is assumed, then this fixed point will approximate the solution of the initial value problem [16]. The proof of these details is similar to those of Theorem 1.

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