CIENCIA 9(4), 433-441, 2001
Maracaibo, Venezuela

Tensorial theory of relativistic quantum mechanics in 1+1 dimensions

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Recibido: 09-06-99 Aceptado: 30-04-01

Abstract

The relativistic quantum mechanical tensorial theory in 1 + 1 dimensions is considered. Its kinematical and dynamical features are reviewed as well as the problem of finding the Dirac spinor for given finite multivectors. For stationary states, the dynamical tensorial equations, equivalent to the Dirac equation, are solved for a free particle and for a particle inside a box.

Key words: 1+1 dimensions; dirac spinor; quantum mechanics; tensorial theory.

Teoría tensorial de la mecánica cuántica relativística en 1+1 dimensiones

Resumen

En este trabajo se considera la teoría tensorial de la mecánica cuántica relativísta en 1+1 dimensiones. Se revisan las caracteriticas cineterísticas cinemáticas y dinámicas de esta teoría, así como tambien se trata el problema de encontrar el espinor de Dirac para multivectores finitos dados. Para estados estacionarios, las ecuaciones tensoriales dinámicas, equivalentes a la ecuación de Dirac, se resuelven para una partícula libre y para una partícula en una caja.

Palabras clave: 1+1 dimensiones; espinor de Dirac; Mecánica cuántica; teoría tensorial.

Introduction

From the beginning of the quantum theory it has been accepted that spinors are essential for describing physical systems in the quantum domain. However, a physical system can be described by giving simultaneously the observables and the state in the form of tensorial densities, that is, probability densities, currents and fields that may be bilinearly defined in terms of the relativistic wave function and its derivatives. Physicists like Pauli, Gordon, Belinfante, Proca (1-4), among others, studied this type of quantities by their utility in the interpretation of relativistic quantum theory. The idea of formulating relativistic quantum mechanics without using spinors, and in the form of a hidrodynamic of tensorial densities which satisfy dynamical equations equivalent to the Dirac equation, was considered by Costa de Beauregard and Takabayasi (5, 6). The inversion of the bilinear relations which permits to express the Dirac spinor in terms of multivectors, has been considered (7). The dynamical aspects of the tensorial theory show the existence of an equivalence theorem between the Dirac equation and

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Maxwell-like equations This allows to visualize relativistic quantum mechanics as a generalized electromagnetism (8). Despite of this, solving problems by using the tensorial theory in 3 + 1 dimensions is troublesome. The reason for that is the non-linear character of the resultant dynamical equations, and the complexity of the probabilistic fluid associated to spin. The tensorial theory in 1 + 1 dimensions is certainly more simple, but as far as we know, it has not been considered in the scientific literature (9). We shall study this theory obtaining a dynamical equation for the probability density, solving this equation for some standard problems, in particular, that of a particle inside a one-dimensional box.

In this paper, the problem of the various boundary conditions that may be imposed for a relativistic "free" particle inside a one-dimensional box will be considered. The spinorial problem of a Dirac fermion in a one dimensional box interacting with a scalar solitonic potential was considered earlier with periodic boundary conditions (10), as well as with more general ones (11), to elucidate the phenomenon of fractional fermion number. For the case of the Dirac "free" massless operator, also in 1 + 1 dimensions, eigenvalues and eigenfunctions were obtained for a family of self-adjoint extensions (12), and the case with a non-zero vector potential was also examined (13). A detailed study of the possible boundary conditions, i.e., self-adjoint extensions, for a relativistic particle inside a box, as well as their non-relativistic limits, has been considered by two of us [V.A. and S. De V.] (14).

In this paper we use the tensorial theory in order to understand the physics behind some of the spinorial boundary conditions that make self-adjoint the "free" hamiltonian for a particle in a box.

In sections I and II, we review the kinematical and dynamical structures of the tensorial theory, as well as the problem of finding the Dirac spinor in terms of finite multivectors. In section III we particularize the obtained results for stationary states. Finally, in sections IV and V we study the free particle and the problem of a particle inside a box.

1. Non-dynamical Structure of the Tensorial Theory

Let $\Psi = \Psi(x,t)$ be a Dirac-like spinor which does not necessarily satisfy the Dirac equation In 1 + 1 dimensions Ψ is a two components spinor and represents the quantum state, but a pure state can also be described by an observable density matrix, that is

$$C = 2\Psi \otimes \overline{\Psi} = 2\Psi\overline{\Psi} \qquad [1]$$

where $\overline{\Psi} = \Psi^{\dagger}\gamma^0$ is the Dirac adjoint of the spinor, $\Psi^{\dagger}$ is the hermitian conjugate spinor and γ^0 is one of the gamma matrices $\gamma^{\mu} = (\gamma^0, \gamma^1) = (\beta, \beta\alpha)$ that satisfy the so called Clifford relation:

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu} = 2\, diag(1,-1)$$

The Clifford number C is a 2×2 complex matrix. The Dirac adjoint of a Clifford number C is defined as $\overline{C} = \gamma^0 C^{\dagger}\gamma^0 = C$ and can be written in a unique way as a linear combination of four matrix basis: $\Gamma^A = \left\{1, \gamma^{\mu}, -i\gamma^5\right\}$ where $\gamma^5 = \alpha$ verifies $\gamma^5\gamma^{\mu} + \gamma^{\mu}\gamma^5 = 0$, that is:

$$C = \sum_A^4 \lambda_A \Gamma^A = S1 + V_{\mu}\gamma^{\mu} - i\overset{+}{\omega}\gamma^5 \qquad [2]$$

The components $\lambda_A = \left\{S, V^{\mu}, \overset{+}{\omega}\right\}$ of C are real and can be obtained by using the scalar product for matrices:

$$\lambda_A = (\Gamma^A, C) = \frac{1}{2}Tr[(\Gamma^A)^{\dagger} C],$$

where the symbol Tr means trace. So:

$$S = \overline{\Psi}\Psi \qquad [3]$$

$$V^{\mu} = \overline{\Psi}\gamma^{\mu}\Psi \qquad [4]$$

$$\overset{+}{\omega} = i\bar{\Psi}\gamma^5\Psi \qquad [5]$$

which are the finite multivectors. We shall refer to V^μ as to the two-vector probability current, S and V^0 as to the internal and the probability densities, respectively, V^1 is the spatial component of the probability current, finally $\overset{+}{\omega}$ is the pseudo-scalar density (7). (The symbol + denotes the dual operation, so $\overset{+}{\omega}$ is a pseudo-tensor).

It is convenient to write explicitely equations [3], [4] and [5] as: $S = \Psi^\dagger\beta\Psi$, $V^0 = \Psi^\dagger\Psi$, $V^1 = \Psi^\dagger\alpha\Psi$ and $\overset{+}{\omega} = i\,\Psi^\dagger\beta\alpha\Psi$. Any of the 2×2 Pauli matrices can be used as the Dirac matrices α and β. In the standard or Dirac representation in 1 + 1 dimensions, $\alpha = \sigma_x$ and $\beta = \sigma_z$. In the so called Weyl representation, $\alpha = \sigma_z$ and $\beta = \sigma_x$.

From equation [1], we get:

$$\left(\frac{C}{2S}\right)^2 = \frac{C}{2S} \qquad [6]$$

that is to say, $\frac{C}{2S}$ represents a pure state.

Substituting equation [2] in [6] and using the identities:

$$\gamma^\mu\gamma^\nu = 1g^{\mu\nu} - \gamma^5\,\overset{+}{D}{}^{\mu\nu},\quad \gamma^\mu\gamma^5 = \overset{+}{D}{}^{\mu\nu}\,\gamma_\nu$$

where $\overset{+}{D}_{\mu\nu} = -\overset{+}{D}_{\mu\nu}$ and $\overset{+}{D}_{01} = 1$ ($\overset{+}{D}_{\mu\nu}$ being the permutation pseudo-tensor), we obtain as a consequence of the purity of the quantum state

$$S^2 + \overset{+}{\omega}{}^2 = V^\mu V_\mu \qquad [7]$$

This relation implies that only three finite multivectors are independents.

Introducing the covariant derivatives of Ψ and $\bar{\Psi}$ by using $D_\mu = \partial_\mu + \frac{ieA_\mu}{\hbar c}$, $\partial_\mu = \left(\frac{1}{c}\partial_{t'},\partial_x\right)$, and $\lambda = \frac{\hbar}{2mc}$, we may write the set of Clifford numbers:

$$C_\mu = 2i\lambda(D_\mu\Psi\otimes\bar{\Psi} - \Psi\otimes\overline{D_\mu\Psi}) \qquad [8]$$

Which may also be written as:

$$C_\mu = I_\mu 1 + T_{\nu\mu}\gamma^\nu - i\,\overset{+}{h}_\mu\,\gamma^5 \qquad [9]$$

The differential multivectors defined by equation [9] are obtained, as we pointed out above, by using the scalar product between matrices. That is

$$\begin{aligned} I_\mu &= i\lambda(\bar{\Psi}D_\mu\Psi - \overline{D_\mu\Psi}\,\Psi) \\ &= i\lambda(\bar{\Psi}\partial_\mu\Psi - \overline{\partial_\mu\Psi}\,\Psi) - \frac{eA_\mu S}{mc^2} \end{aligned} \qquad [10]$$

$$\begin{aligned} T_{\mu\nu} &= i\lambda(\bar{\Psi}\gamma_\mu D_\nu\Psi - \overline{D_\nu\Psi}\gamma_\mu\,\Psi) \\ &= i\lambda(\bar{\Psi}\gamma_\mu\partial_\nu\Psi - \overline{\partial_\nu\Psi}\gamma_\mu\,\Psi) - \frac{eA_\nu V_\mu}{mc^2} \end{aligned} \qquad [11]$$

$$\begin{aligned} \overset{+}{h}_\mu &= i\lambda(i\bar{\Psi}\gamma^5 D_\mu\Psi - i\overline{D_\mu\Psi}\gamma^5\,\Psi) \\ &= i\lambda(i\bar{\Psi}\gamma^5\partial_\mu\Psi - i\overline{\partial_\mu\Psi}\gamma^5\,\Psi) - \frac{eA_\mu\overset{+}{\omega}}{mc^2} \end{aligned} \qquad [12]$$

where $e = -|e|$ is the electron charge and m is their mass. The electromagnetic potential is $A^\mu = (V, A)$. Obviously we may choose $A = 0$ in 1 + 1 dimensions.

The differential multivectors I_μ and $\overset{+}{h}_\mu$ are called respectively, the convective current and pseudo-current and $T_{\mu\nu}$ the probability tensor. By constructing the quantities $CC_\mu \pm C_\mu C$ the following relation between multivectors may be obtained

$$SI^\mu + \overset{+}{\omega}\,\overset{+}{h}{}^\mu = V_\nu T^{\nu\mu} \qquad [13]$$

$$\overset{+}{D}_{\mu\nu}(V^\nu\,\overset{+}{h}_\rho - \overset{+}{\omega}\,T_\rho{}^\nu) = \lambda(V_\mu\partial_\rho S - S\partial_\rho V_\mu) \qquad [14]$$

$$\overset{+}{D}_{\mu\nu}\,V^\mu T_\rho{}^\nu = \lambda(S\,\partial_\rho\,\overset{+}{\omega} - \overset{+}{\omega}\,\partial_\rho S) \qquad [15]$$

$$\lambda \overset{+}{D}_{\mu\nu} V^{\mu} \partial_{\rho} V^{\nu} = \overset{+}{\omega} I_{\rho} - S \overset{+}{h}_{\rho} \quad [16]$$

$$\lambda \overset{+}{D}_{\mu\nu} (V^{\nu} \partial_{\rho} \overset{+}{\omega} - \overset{+}{\omega} \partial_{\rho} V^{\nu}) = S T_{\mu\rho} - I_{\rho} V^{\mu} \quad [17]$$

2. Dynamical Structure of the Tensorial Theory

Dynamical equations

Let $O_D = i\gamma^{\mu} D_{\mu} - \frac{mc}{\hbar}$ be the Dirac operator. If Ψ satisfies the Dirac equation:

$$O_D \Psi = 0 \quad [18]$$

Ψ becomes a Dirac spinor.

Let us define the Clifford number:

$$C_D = O_D \Psi \otimes \overline{\Psi} \quad [19]$$

In view of equation [23], each complex multivector belonging to C_D is null, that is:

$$\overline{\Psi} \Gamma^{A} O_D \Psi = 0 \quad [20]$$

Thus, the real and imaginary parts of equations [20] are zero. Making full use of the definitions of the multivectors and of the relations between the gamma matrices given above, in addition to the following: $\gamma^{\mu\dagger} = \gamma^{0}\gamma^{\mu}\gamma^{0}$; $\gamma^{5\dagger} = \gamma^{5}$ and $\overset{+}{D}^{\mu\nu} \overset{+}{D}_{\alpha\beta} = -\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} + \delta^{\mu}_{\beta}\delta^{\nu}_{\alpha}$ we obtain the dynamical equations implied by the Dirac equation These equations may be conveniently grouped in three pairs:

$$\lambda \overset{+}{D}^{\mu\nu} \partial_{\nu} \overset{+}{\omega} = I^{\mu} - V^{\mu} \quad [21]$$

$$\lambda \overset{+}{D}^{\nu\mu} \partial_{\nu} S = \overset{+}{h}^{\mu} \quad [22]$$

$$T^{\mu}_{\mu} = S \quad [23]$$

$$\overset{+}{D}^{\mu\nu} \overset{+}{T}_{\mu\nu} = 0 \quad [24]$$

$$\partial_{\mu} V^{\mu} = 0 \quad [25]$$

$$\lambda \overset{+}{D}^{\mu\nu} \partial_{\mu} V^{\nu} = \overset{+}{\omega} \quad [26]$$

These equations are the fundamental dynamical relations implied by the Dirac equation

The currents I^{μ}, $\overset{+}{h}^{\mu}$ and V^{μ} in equation [25], are conserved inasmuch as each one of them satisfies a continuity equation

The above three pairs of dynamical equations [21-26] look like a rather large set of relations to be satisfied. However, the first pair, which we call the Maxwell-like equations, may be considered as the fundamental one. In fact, the other two pairs may be derived from these Maxwell-like equations and the non-dynamical relations [7, 13, 14].

So, the dynamical information of the Dirac equation is contained in only two vectorial equations In other words, if the Dirac equation is verified, it implies three pairs of dynamical equations which with help of the non-dynamical or algebraic relations may be reduced to a single pair. On the other hand, if the three pairs of dynamical relations are verified, then [20] is satisfied for every Γ^{A}. The spinor $\overline{\Psi}$ is in general not null, thus Dirac equation is satisfied. In this way, the equivalence between Maxwell-like equations and the Dirac equation is completed.

Let us assume that we know the necessary multivectors and that we want to obtain their corresponding spinor. In the next section we will show how to obtain the spinor for a given set of finite multivectors.

Spinors from multivectors

Let us first consider the following general spinor in the Dirac representation

$$\Psi(x,t) = \begin{pmatrix} \sqrt{a}\, e^{i(\varepsilon+\Omega)} \\ \sqrt{b}\, e^{i(\varepsilon-\Omega)} \end{pmatrix} \quad [27]$$

From equations [3,4,5] one obtains: $S = a - b$, $V^0 = a + b$, $V^1 = 2\sqrt{abcos}(2\Omega)$ and $\overset{+}{\omega} = 2\sqrt{absin}(2\Omega)$. From which one gets

$$a = \frac{1}{2}(V^0 + S) \qquad [28]$$

$$b = \frac{1}{2}(V^0 - S) \qquad [29]$$

$$cos(2\Omega) = \frac{V^1}{\left[(V^1)^2 + (\overset{+}{\omega})^2\right]^{\frac{1}{2}}} \qquad [30]$$

In order to obtain the overall phase ε up to an integration constant, one may use any of the Maxwell-like equations It is convenient to write them only in terms of finite multivectors, the external electromagnetic potential A_μ and the gradient of ε. For this, let us write the currents I^μ and $\overset{+}{h}{}^\mu$ in terms of the spinor [27]:

$$I_\mu = -2\lambda(S\,\partial_\mu \varepsilon + V^0 \partial_\mu \Omega) - \frac{eA_\mu S}{mc^2} \qquad [31]$$

$$\overset{+}{h}_\mu = -2\lambda\,\overset{+}{\omega}(\partial_\mu \varepsilon + \frac{V^0}{S}\partial_\mu \Omega)$$

$$-\frac{eA_\mu \overset{+}{\omega}}{mc^2} - \frac{\lambda}{S}\overset{+}{D}{}^{\alpha\rho}\, V_\alpha \partial_\mu V_\rho\, 4 \qquad [32]$$

where we have used equation [16] in order to express $\overset{+}{h}{}^\mu$ in term of I_μ.

Substituting equation [31] in [21] and [32] in [22], one gets:

$$\lambda\,\overset{+}{D}_{\mu\nu}\,\partial^\nu\,\overset{+}{\omega} = -2\lambda(S\,\partial_\mu \varepsilon + V^0 \partial_\mu \Omega)$$

$$-\frac{eA_\mu S}{mc^2} - V_\mu \qquad [33]$$

$$\lambda S\,\overset{+}{D}_{\mu\nu}\,\partial^\nu S = \lambda\,\overset{+}{D}{}^{\alpha\rho}\, V_\alpha \partial_\mu V_\rho + 2\lambda\,\overset{+}{\omega}(S\,\partial_\mu \varepsilon + V^0 \partial_\mu \Omega) +$$

$$\frac{eA_\mu S\,\overset{+}{\omega}}{mc^2} \qquad [34]$$

In this way, with equations [33] or [34] both implied by the Dirac equation, one obtains $\partial_\mu \varepsilon$ from the finite multivectors and the external electromagnetic potential. This quantity is observable. Integrating it, one can calculate ε up to an integration constant.

3. Stationary States

In the case of stationary states the overall phase may be written as: $\varepsilon(x,t) = \varepsilon = -\frac{E}{\hbar}t + f(x)$, moreover $\partial_t \Omega = 0$ and $\partial_t A_\mu = 0$. So, from the general spinor in the Dirac representation [27], one obtains $\Psi(x,t) = \psi(x)e^{-i\left(\frac{E}{\hbar}\right)t}$, where $\psi(x) = \begin{pmatrix} \sqrt{a}\, e^{i(f+\Omega)} \\ \sqrt{b}\, e^{i(f-\Omega)} \end{pmatrix}$.

The finite multivectors may be written as:

$$S = \phi^* \phi - \chi^* \chi$$
$$V^0 = \phi^* \phi + \chi^* \chi$$

$$V^1 = \phi^* \chi + \chi^* \phi$$
$$\overset{+}{\omega} = i(\phi^* \chi - \chi^* \phi)$$

where $\phi = \sqrt{a}\, e^{i(f+\Omega)}$ and $\chi = \sqrt{b}\, e^{i(f-\Omega)}$ are respectively the spatial parts of the so called large and small components of the spinor Ψ in the Dirac representation.

Denoting hereafter with primes the differentiation with respect to x and choosing the so called axial gauge $A^1 = 0$ and $eA_0 \equiv U$ we can write the following relevant Maxwell-like equations

$$-\lambda\,\overset{+}{\omega}' = \left(\frac{E - U}{mc^2}\right)S - V^0 \qquad [35]$$

$$\lambda S' = \left(\frac{E-U}{mc^2}\right)\overset{+}{\omega} \qquad [36]$$

The other dynamical equations implied by the Maxwell-like equations are

$$(V^1)' = 0 \qquad [37]$$

$$\lambda(V^0)' = \overset{+}{\omega} \qquad [38]$$

We emphasize that the pseudo scalar $\overset{+}{\omega}$ is nothing but the "scaled gradient" of the probability density V^0. In addition to these relations, we have the kinematical relation [7]:

$$S^2 + \overset{+}{\omega}^2 = (V^0)^2 - (V^1)^2$$

Returning to the problem of calculating the spinor starting from multivectors for stationary states, we obtain from the spatial component of equation [33] a differential equation that permits us to get the phase $f(x)$ up to an integration constant, with Ω obtained from equation [30]:

$$2\lambda(Sf' + V^0\Omega') = V^1 \qquad [39]$$

The time component of equation [33] is precisely equation [35].

Given a potential energy $U(x)$, the Maxwell-like equations yield a linear differential equation for the probability density. By using equations [38, 35, 36] we obtain:

$$(V^0)''' + g(V^0)'' + (2k)^2(V^0)' - \frac{g}{\lambda^2}V^0 = 0 \qquad [40]$$

where:

$$g(x) = \frac{U'(x)}{E-U(x)} \text{ and } k^2 = \frac{(E-U)^2 - (mc^2)^2}{\hbar^2 c^2}.$$

It is worth mentioning that obtaining a dynamical equation for the probability density has been from the beginnings an important aim of quantum mechanics.

In the next section we solve this equation for $g(x) = 0$ in this case a harmonic type equation is obtained.

4. Free Particle

Let us now solve the Maxwell-like equations for a free particle. Making $U = 0$ in the equation [40], we obtain with $k^2 = \frac{E^2 - (mc^2)^2}{\hbar^2 c^2}$, the fundamental equation for the probability density $V^0(x)$:

$$(V^0)''' + (2k)^2(V^0)' = 0 \qquad [41]$$

The general solution of equations [41, 37] may be written as:

$$V^0(x) = \frac{mc^2}{\hbar ck}[A + B\sin(2kx) - C\cos(2kx)] > 0 \qquad [42]$$

$$V^1(x) = D \qquad [43]$$

where A, B, C and D are arbitrary constants. For the other two finite multivectors the general solution is given by:

$$S(x) = \frac{E}{mc^2}V^0(x) - \frac{\hbar ck}{E}A \qquad [44]$$

$$\overset{+}{\omega}(x) = B\cos(2kx) + C\sin(2kx) \qquad [45]$$

Using the constraint [7] one obtains:

$$-\frac{(mc^2)^2}{E^2}A^2 + B^2 + C^2 + D^2 = 0 \qquad [46]$$

The most simple solution of equation [42] is a constant probability density:

$$V^0(x) = \frac{mc^2}{\hbar ck}A > 0 \qquad [47]$$

Thus, equations [44,45] can be written as:

$$S(x)=\frac{(mc^2)^2}{\hbar ckE}A \qquad [48]$$

$$\overset{+}{\omega}(x)=0 \qquad [49]$$

The probability current is given by:

$$V^1(x)=\pm\frac{mc^2}{E}A \qquad [50]$$

where the constraint [46] has been used. Obviously, A is a normalization constant.

The Dirac spinor [27], that is $\Psi(x)=\begin{pmatrix}\sqrt{a}\,e^{i(f+\Omega)}\\ \sqrt{b}\,e^{i(f-\Omega)}\end{pmatrix}e^{-i\frac{E}{\hbar}t}$, may be obtained by using equations [28, 29, 30, 39], from which one obtains:

$$\Psi\propto\sqrt{\frac{mc^2}{\hbar ck}}A\begin{pmatrix}1\\ \pm\dfrac{\hbar ck}{E+mc^2}\end{pmatrix}e^{\pm ikx}e^{-i\frac{E}{\hbar}t} \qquad [51]$$

This spinor is the well known solution to the Dirac equation for a free particle.

5. "Free" Particle Inside a Box

Let us now consider a relativistic "free" particle confined inside a one-dimensional box with fixed walls at $x=0$ and $x=L$. In order to obtain the four arbitrary constants and the energy eigenvalues, instead of considering a confinement potential at the walls of the box, we impose upon the solutions [42-45] adequate boundary conditions.

Using equation [43], the constraint [7] becomes

$$(V^0)^2(x)-S^2(x)-\overset{+}{\omega}{}^2(x)=D^2 \qquad [52]$$

For a particle confined inside a box, we put $V^1(0)=V^1(L)=0$, then $D=0$ everywhere and

$$S^2(x)+\overset{+}{\omega}{}^2(x)=(V^0)^2(x) \qquad [53]$$

So, $V^0(x)$ cannot vanish unless $S(x)=\overset{+}{\omega}(x)=0$ but this yields the trivial solution.

Using equation [53] at the boundaries of the box.

$$S^2(0)+\overset{+}{\omega}{}^2(0)=(V^0)^2(0)$$

$$S^2(L)+\overset{+}{\omega}{}^2(L)=(V^0)^2(L) \qquad [54]$$

In order to satisfy this set of relations, one may write for $0\le\theta,\xi<2\pi$

$$S(0)=-cos\theta\ V^0(0),\ \overset{+}{\omega}(0)=-sin\theta\ V^0(0) \qquad [55]$$

$$S(L)=-cos\xi\ V^0(L),\ \overset{+}{\omega}(L)=-sin\xi\ V^0(L) \qquad [56]$$

where the parameters θ,ξ label the subfamilies of boundary conditions. It can be shown that this two parameters family of boundary conditions is the most general one for a particle confined in a box, and that they are included in the domain of the Dirac Hamiltonian for a "free" particle inside a box. We will only consider those boundary conditions that are symmetrical under space inversions. The fundamental equation [41] is invariant under space inversions if

$$V^0(x)=V^0(L-x) \qquad [57]$$

Then, by using equations [38,44], we obtain

$$\overset{+}{\omega}(x)=-\overset{+}{\omega}(L-x) \qquad [58]$$

$$S(x)=S(L-x) \qquad [59]$$

In this case we only have a one-parameter family

$$S(0)=cos\ \xi\ V^0(0),\ \overset{+}{\omega}(0)=sin\ \xi\ V^0(0) \qquad [60]$$

$$S(L) = \cos\xi \; V^0(L),\; \overset{+}{\omega}(L) = -\sin\xi \; V^0(L) \qquad [61]$$

Among the infinite boundary conditions parametrized by ξ we choose the most simple ones $\xi = 0, \pi$ and $\xi = \pi/2,\ 3\pi/2$. In the following list we specify the tensorial boundary conditions for the local observables [TBC], their corresponding spinorial boundary conditions [SBC], and the energy eigenvalue equations [EEE]. For the first case $\xi = 0, \pi$ we obtain:

$$\overset{+}{\omega}(0) = \overset{+}{\omega}(L) = 0 \qquad [62]$$

a) TBC: $S(0) = -V^0(0),\ \ S(L) = -V^0(L)$ [63]
SBC: $\phi(0) = \phi(L) = 0$
EEE: $\cos(2kL) = 1$

b) TBC: $S(0) = V^0(0),\ \ S(L) = V^0(L)$ [64]
SBC: $\chi(0) = \chi(L) = 0$
EEE: $\cos(2kL) = 1$

For the second case $\xi = \pi/2,\ 3\pi/2$, we obtain:

$$S(0) = S(L) = 0 \qquad [65]$$

c) TBC: $\overset{+}{\omega}(0) = -V^0(0),\ \ \overset{+}{\omega}(L) = V^0(L)$ [66]
SBC: $\chi(L) = -i\phi(L),\ \ \chi(0) = i\phi(0)$
EEE: $\tan(kL) - \dfrac{\hbar c k}{mc^2} = 0$

d) TBC: $\overset{+}{\omega}(0) = V^0(0),\ \ \overset{+}{\omega}(L) = -V^0(L)$ [67]
SBC: $\chi(L) = i\phi(L),\ \ \chi(0) = -i\phi(0)$
EEE: $\tan(kL) + \dfrac{\hbar c k}{mc^2} = 0$

Then, using the constraint [7] the multivectors for the first case are:

$$V^0(x) = A\,\frac{mc^2}{\hbar c k}\left[\pm 1 - \frac{mc^2}{E}\cos(2kx)\right] \qquad [68]$$

$$V^1(x) = 0 \qquad [69]$$

$$S(x) = \frac{E}{mc^2}V^0(x) - \frac{\hbar c k}{E}A \qquad [70]$$

$$\overset{+}{\omega}(x) = A\,\frac{mc^2}{E}\sin(2kx) \qquad [71]$$

The upper sign corresponds to the boundary condition a) and the lower one to the boundary condition b) Note that in order that $V^0(x)$ be positive the lower sign in equation [68] must be used only for electrons with negative energy.

The multivectors for the second case are:

$$V^0(x) = A\,\frac{(mc^2)^2}{E^2}\left[\frac{E^2}{\hbar c k mc^2} \pm \sin(2kx) - \frac{mc^2}{\hbar c k}\cos(2kx)\right] \qquad [72]$$

$$V^1(x) = 0 \qquad [73]$$

$$S(x) = \frac{E}{mc^2}V^0(x) - \frac{\hbar c k}{E}A \qquad [74]$$

$$\overset{+}{\omega}(x) = A\,\frac{(mc^2)^2}{E^2}\left[\sin(2kx) \pm \frac{\hbar c k}{mc^2}\cos(2kx)\right] \qquad [75]$$

Where the upper sign corresponds to the boundary condition c) and the lower one to the boundary condition d).

Knowing the finite multivectors, the Dirac spinors may be obtained using the relations [28-30] and [39]. For the first case of boundary conditions the followiing spinors are obtained for a) and b).

$$\Psi \propto \sqrt{A}\begin{pmatrix} \sin(kx) \\ \dfrac{-i\hbar c k}{E + mc^2}\cos(kx) \end{pmatrix} e^{-i\frac{E}{\hbar}t} \qquad [76]$$

$$\Psi \propto \sqrt{A}\begin{pmatrix} \cos(kx) \\ \dfrac{i\hbar c k}{E + mc^2}\sin(kx) \end{pmatrix} e^{-i\frac{E}{\hbar}t} \qquad [77]$$

where the upper and the lower sign respectively has been used in order to obtain the spinors [76,77] respectively.

For the second case of boundary conditions $\xi = \pi/2,\ 3\pi/2$, using the upper sign of the equations [72,75], we obtain the spinor:

$$\Psi \propto \sqrt{A}\begin{pmatrix} sin(kx - \frac{\zeta}{2}) \\ \frac{-i\hbar c k}{E + mc^2} cos(kx - \frac{\zeta}{2}) \end{pmatrix} e^{-i\frac{E}{\hbar}t} \qquad [78]$$

where $tan(\zeta) = \frac{\hbar c k}{mc^2}$. Using the lower sign we obtain:

$$\Psi \propto \sqrt{A}\begin{pmatrix} cos(kx - \frac{\delta}{2}) \\ \frac{i\hbar c k}{E + mc^2} sin(kx - \frac{\delta}{2}) \end{pmatrix} e^{-i\frac{E}{\hbar}t} \qquad [79]$$

where $tan(\delta) = -\frac{\hbar c k}{mc^2}$.

Conclusions

We have seen that from the tensorial viewpoint the relativistic quantum mechanics in 1 + 1 dimensions appears as a very rich theory, whose objects are explicitly local observables which satisfy Maxwell-like equations In contrast, in the spinorial formulation, which is very much compact and relatively simple, the relevant information is implicit and the physical meaning of all its objects is not straightforward. In the tensorial theory, we have a linear differential equation for the observable probability density which is a consequence of the Maxwell-like equations equivalent to the Dirac equation Then, imposing symmetries and boundary conditions is easier, and is more physically meaningfull. For example, the current V^{μ} cannot vanish anywhere, inasmuch as in that case, the other two finite multivectors, the scalar and the pseudo scalar, should also be zero. The only possibility is the vanishing of the spatial component of V^{μ}. If $V^{1} = 0$ at a boundary, either the scalar or the pseudo scalar may vanish, but not both of them.

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