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Spectral properties of Levy matrices*

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Abstract

The statistical properties of the spectrum from large symmetric matrices are investigated. In these matrices the elements are chosen from a power-law distribution $p(x) = |\nu| x^{\nu-1}$ with $-2 \le \nu \le 1$. Universality classes or stable laws are explored by studying the density of states ρ(E) and the distribution of eigenvalue spacings P(s). Various regimes could be identified as a function of the disorder strength parameter ν. For $0 < \nu < 1$, the density of states obeys the simple semicircular law, and P(s) follows the Wigner surmise. For $\nu < 0$, various zones separated by energy-dependent boundaries are observed. Furthermore, in the limit $\nu \to 0$, we find a density of states that corresponds to the sparse matrix limit, with the characteristic singularity $\langle \rho(E) \rangle \propto 1/E$. However, in this limit the spacing distribution exhibits power laws tails instead of the well known Brody form.

Key words: Density of states; energy levels; Random matrices; sparse matrices.

Propiedades espectrales de matrices de Levy

Resumen

Se investigan las propiedades estadísticas del espectro de matrices simétricas grandes cuyos elementos son elegidos de distribuciones tipo ley de potencia $p(x) = |\nu| x^{\nu-1}$, con $-2 \le \nu \le 1$. Se exploran las clases de universalidad o leyes estables a través del estudio de la densidad de estados ρ(E) y la distribución de espaciamiento de los autovalores P(s). Como función del parámetro de desorden ν se identifican varios regímenes. Para $0 < \nu < 1$, la densidad de estados sigue la ley semicircular, y P(s) la premisa de Wigner. Para $\nu < 0$, se observan varias fases separadas por fronteras que dependen de la energía. Además, en el límite $\langle \rho(E) \rangle \propto 1/E$, se encuentra una densidad de estados que corresponde al límite de matrices ralas con la singularidad característica . Sin embargo, en éste límite, la distribución de espaciamientos exhibe colas algebraicas en lugar de ajustarse a la forma clásica de Brody.

Palabras clave: Densidad de estados; matrices aleatorias; matrices ralas; niveles de energía.

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1. Introduction

Random matrices (RM) have been investigated intensively in the last decade, due to the wide range of applications to different branches of physics such as the theory of mesoscopic fluctuations in disordered conductors (1), some spin-glass models (2), light propagation in dense media (3), and quantum chaos, (4) among others. The simplest case of RM is the one where the elements of a symmetric matrix are identically distributed random variables with zero mean and finite variance. It has been shown that in the limit of large matrices, the distribution of eigenvalue spacings obeys the Wigner surmise i.e., a universal form (or stable law for the spectra) which depends only on symmetry and reflects strongly correlated eigenvalues due to level repulsion (5). These matrices belong to the so called Gaussian orthogonal ensemble (GOE).

Here, we consider large NxN symmetric matrices whose elements H_{ij} ($=H_{ji}$) are independent random variables chosen from a power law distribution

$$p(H) = |\nu||H|^{\nu-1} \quad [1]$$

with $-2 \leq \nu \leq 1$. The power law distribution, *p(H),* permits testing the limits of known stable laws for eigenvalue spectra. These matrices, known as Levy matrices when $-2 \leq \nu \leq 0$ (6), display a variety of behaviors as a function of ν, as will be shown in the following sections. The limit $\nu \to 0$ of particular interest here, since it behaves similar to an ensemble of sparse random matrices which do not belong to the Gaussian universality class. Sparse Random Matrices (SRM) are constructed by assigning a fraction p/N of non-zero elements per matrix row. The distribution of eigenvalue spacings for SRM´s is best fitted to a Brody form (7). It is easy to imagine that the structure of SRM is composed of fractal structures which, depending on the fraction p/N, can be isolated clusters or form a *connected* spanning clusters (always obeying matrix symmetries). This behavior is related with its singular spectral properties. In the same fashion one can talk about an effective "connectivity" of large Levy matrices as a function of the parameter ν. When $\nu = 1$, most matrix elements are comparable in size, and the matrix could be seen as a connected block of dimension N (infinite cluster). As ν decreases (for $\nu > 0$), more and more elements become very small, and the matrix can be separated into blocks of finite size. Thus, in terms of ν there is a percolation like process associated with crucial system changes. The disintegration of the matrix into finite blocks occurs at a well defined value of $\nu = \nu_c$, which is equivalent to the concept of percolation threshold p_c. Similarly, the "mean connectivity" parameter p can be defined as a continuous function of ν* (8). The semicircular law is recovered in the limit where the mean "connectivity" p gets close to the threshold i.e., the mean number of non-zero elements per row of the SRM tends to infinity. For any $p < p_c$, however, there are states beyond the semicircle. SRM are intimately related to interesting problems such as Anderson model on the Bethe lattice with a random Poisson-like local branching number (9), and are useful to study dilute spin systems (10).

Some comments on the properties of *p(H)* of equation [1] are in order: For $\nu < 0$, the matrix elements take values $1 < |H_{ij}| < \infty$. The distribution *p(H)* has finite variance and mean for $\nu < -2$, this parameter range should give rise to the classical GOE limit. For $-2 < \nu \leq 0$, *p(H)* has diverging variance, and

* The approach to this problem by critical path analysis similar to the used by R.F. Angulo and E. Medina in J. Stat. Phys. 75:135, 1994 will be published elsewhere.

new spectral properties are expected. For $0 < \nu < 1$, the matrix elements are confined to the interval $0 < |H_{ij}| < 1$, and thus both the mean and variance are always finite. Nevertheless, interesting phenomena is seen in the limit $\nu \to 0^+$, where a hierarchy of matrix elements sizes arises with long tails towards small elements. We perform numerical simulations diagonalizing matrices for sizes up to N = 2025. The IMSL Fortran subroutines were used to calculate the eigenvalues and eigenvectors of such matrices. Samples of ~ 400 matrices were typically generated to get good statistical data. Numerical eigenvalues are analyzed to compute the density of states and the distribution of eigenvalue spacings presented in this paper.

The outline of this paper is as follows. In Section 2 we present results on the density of states. Section 3 is dedicated to the study of the distribution of eigenvalue spacings, and we end with the conclusions.

2. Density of States

The average density of states of the matrix spectrum for Gaussian ensembles of RM is a universal function. Its limit form for large matrices is independent of the disorder of the matrix elements (they always have finite variance), and whether the symmetry is orthogonal, unitary, or symplectic. Changes in this function should certainly signal new spectral behavior. For positive ν values ($0<\nu<1$), *p(H)* has finite mean and variance. We find that for $\nu > \nu_c \sim 0.01$ the density of states obeys the semicircular law

$$\langle \rho(E) \rangle = \frac{1}{2\pi}\left(4 - \frac{E^2}{\sigma}\right)^{1/2}, \qquad [2]$$

where σ is the variance (Figure 1). This is exactly the form observed for Gaussian ensembles. However, as the value of $\nu<0.01$, $\rho(E)$ develops deviations from the semicircular law of equation (2). A sharp singularity of $1/|E|$ develops as $|E| \to 0$ as depicted in the sequence of plots for various ν shown in Figure 1. Furthermore, additional singularities arise around E ~ 1. A similar $1/|E|$ singu-

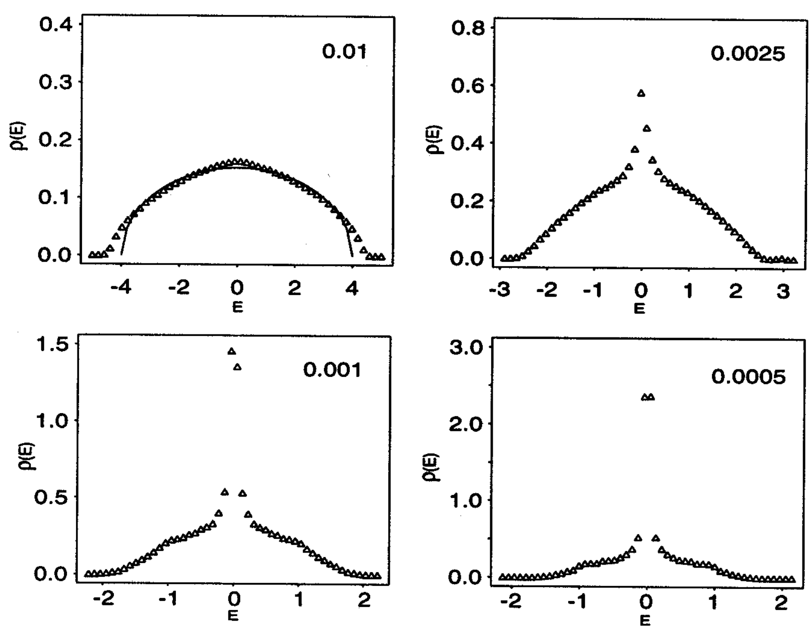

Figure 1. Density of states for matrices of size N = 1600 and the following sequence of ν values (from top to bottom): ν = 0.01, 0.0025, 0.001, and 0.0005 diplaying the appearance of the $1/|E|$ singularity.

lar behavior was previously shown in SRM with matrix elements from a distribution with finite variance* (11). Nevertheless, this behavior renormalizes as a function of N and finally converges to the Gaussian limit for $\nu \neq 0$.

For the range $-2 < \nu \leq 0$, the average density of states $\langle \rho(E) \rangle$ is found to be symmetric around E = 0 and not bounded in the energy range. Cizeau and Bouchaud (6) determined analytically the behavior of $\langle \rho(E) \rangle$ in the limits of small and large energies by using a recursion method. They found that for $E \to 0$ that

$$\langle (E) \rangle \cong a(\nu) - b(\nu) E^2, \qquad [3]$$

where $a(\nu)$ and $b(\nu)$ are parameters that depend on ν. Whereas for large energies

$$\langle \rho(E) \rangle \cong c(\nu) \,/\, E^{1-\nu} \qquad [4]$$

Our numerical data confirm these expressions. The behavior for large energies is determined from a double logarithmic plot of the density and the extrapolation of these values as a function of the matrix size to infinity.

3. Distribution of Eigenvalues Spacings

The unfolded eigenvalue spacing is defined as the difference between successive eigenvalues normalized by its average value, i.e., $S_\alpha = (E_{\alpha+1} - E_\alpha) / \langle S_\alpha \rangle$. From the numerical distributions of *P(s)* we find different behaviors as function of the energy E and ν. For $0.01<\nu<1$, the spacing distribution has the classical Wigner form

$$P_\beta = c_\beta s^\beta \exp\left(-s^2 / w_\beta^2\right), \qquad [5]$$

with β a parameter related to the symmetry of the system. c_β y w_β are constants that also depends on the system's symmetry. For the GOE $\beta = 1$, and $c_1 = \pi/2$, $w_1 = (\pi/4)^{1/2}$. For smaller values of ν our numerical data is not well fitted by this form. We observed that as ν decreases towards zero, the distribution deviates more and more from the Wigner form. In fact, the tail of the distribution is power law in this regime, as suggested in Figure 2b (although less than a decade is sampled). There is also level repulsion for small spacings. From Figure 2a we see that the level repulsion increases as ν decreases. These results differ from those found in the regular sparse random matrix ensemble where the distribution is well fitted to an exponential based distribution, commonly known as Brody distribution (7)

$$P_\sigma{}^B(s) = a s^\sigma \exp\left(-b s^{\sigma+1}\right), \qquad [6]$$

with $a = (\sigma+1)b$, and $b = \left(\Gamma(\sigma + 2/\sigma + 1)\right)^{\sigma+1}$. However, the Brody form has no rigorous justification.

For small spacings, we evaluate the slope of the straight line that best fit the data in the double logarithmic plot of *P(s)* vs. *s*, and found a ν-dependent value. This slope gets smaller as ν decreases. We propose the following simple form for the spacing distribution, *P(s)*, in this regime $P(s) \approx F(s) / s^\mu$ with $F(s) \cong s^\tau$ for $s \to 0$, and $F(s) \cong const$ *for* $s >> 1$ *and* $\tau > \mu$.

When ν is negative, the limit distribution appears to be energy dependent. We find that $-1 < \nu < 0$, *P(s)* is of Wigner type for

* Since, the sparse limit is observed for very small values of ν where the distribution has very strong tails towards zero, one may argue that the matrix elements used as input to the IMSL routines are just zeros in this limit. We checked our algorithms to guarantee that the computer was able to read and process those matrix elements as nonzero values.

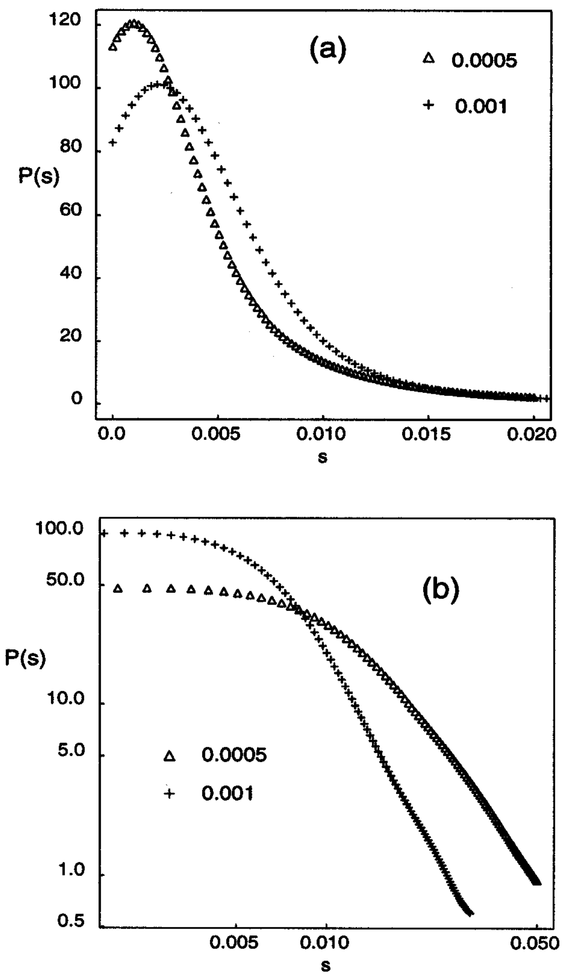

Figure 2. Eigenvalue spacing distribution for matrices in the sparse regime. Data are for matrices of size N = 2025 with ν = 0.001 (+) and 0.0005 (Δ). a) linear and b) double logarithmic scale.

energies less than a critical value $E < E_{c1}$, whereas for energies $E > E_{c1}$, the distribution has exponential form, i.e., it is of Poisson type. This change in the form of *P(s)* reflects the existence of a transition from extended (GOE) states to localized (Poissonian) states. The critical value E_{c1} is a function of ν, which is not trivial to determine.

For $-2 < \nu < -1$, we also find a critical energy E_{c2} that separates two regions where the spacing distribution has different forms. For $E < E_{c2}$, the distribution is very close to the Wigner form, and it is non-universal for $E > E_{c2}$ depending continuously on E. In this regime the distribution has an intermediate form between Wigner and Poisson, and becomes pure Poissonian for large energies. The range of negative ν, has also been studied numerically studied by Cizeau and Bouchaud (6).

4. Conclusions

We have shown that the ensemble of Lévy matrices display a variety of behaviors ranging from the classical GOE (characteristic of extended states) to Poissonian (for localized states) according to the value of the exponent ν of the power law distribution from which the matrix elements are chosen.

For positive ν values, the density of states perfectly satisfy the semicircle law down to a $\nu_c \sim 0.01$, and a crossover to a density of states with a $1/|E|$ singularity for $|E|$ near zero is seen as $\nu \to 0^+$. A similar behavior is seen for GOE matrices in the sparse limit. For negative ν values, the density comprises the whole energy axis and presents long ν-dependent tails for large energies.

The eigenvalue spacing distribution *P(s)*, has also different forms depending on the ν value. For $\nu > 0$, *P(s)* changes from the classical Wigner form to a power law form in the sparse matrix regime. Negative ν values also display interesting behaviors, with distributions which are also energy dependent. Thus we are able to see a transition from GOE Wigner distribution (for small energies) to a Poisson like one (at large E), and a "mixed" region in the transition zone. We have also verified the existence of this transition following Cizeau and Bouchaud´s (6) suggestion of using the matrix eigenvectors to compute the inverse participation ratio that readily indicates if the states are extended or localized.

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